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Bacharelado em Ciência da Computação

Hilbert calculi for the main fragments of Classical Logic

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Hilbert calculi for the main fragments of Classical Logic

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To my mother Adalcina, my father Jezuzeni, my brother Davi, my dear Geovanna, and in memory of Patrik, my eternal friend.

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If you would be a real seeker after truth, it is necessary that at least once in your life you doubt, as far as possible, all things.

René Descartes

Cálculos de Hilbert para os principais fragmentos da Lógica Clássica

Autor: Vitor Rodrigues Greati
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Resumo

A Lógica Clássica, sob a ótica da Álgebra Universal, pode ser vista como aquela induzida pelo clone completo sobre o conjunto $\{0, 1\}$. Os demais clones sobre o mesmo conjunto induzem, portanto, sublógicas ou fragmentos da Lógica Clássica. Em 1941, Emil Post apresentou a organização de todos clones sobre $\{0, 1\}$ em um reticulado ordenado por inclusão [10]. Em [11], Wolfgang Rautenberg explorou esse reticulado para demonstrar que todos esses fragmentos são fortemente e finitamente axiomatizáveis. Rautenberg utilizou uma notação pouco usual e a sobrecarregou diversas vezes, gerando confusão, além de ter apresentado demonstrações incompletas e cometido vários erros tipográficos, imprecisões e desacertos. Em especial, os principais fragmentos da Lógica Clássica — expressão aqui utilizada para se referir àqueles dos quais tratam as demonstrações dos casos principais apresentadas por Rautenberg na primeira parte de seu artigo — merecem uma apresentação mais rigorosa e acessível, pois produzem importantes discussões e resultados sobre os demais clones, além de embasarem os procedimentos recursivos da segunda parte da demonstração do teorema da axiomatizabilidade de todos os clones bivalorados. Neste trabalho, propõe-se uma reapresentação das demonstrações para esses fragmentos, desta vez com uma notação mais moderna, com maior preocupação com os detalhes, com mais atenção à corretude da escrita e com a inclusão de todas as axiomatizações dos clones investigados. Além disso, os sistemas formais envolvidos serão especificados na linguagem do assistente de demonstração Lean, e as demonstrações de completude serão verificadas com a ajuda dessa ferramenta. Dessa forma, a demonstração do resultado apresentado por Rautenberg estará apresentada de forma mais acessível, compreensível e confiável para a comunidade.

Palavras-chave: fragmentos da Lógica Clássica, cálculos de Hilbert, reticulado de Post, Álgebra Universal

Hilbert calculi for the main fragments of Classical Logic

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Abstract

Classical logic, under a universal-algebraic consequence-theoretic perspective, can be defined as the logic induced by the complete clone over $\{0, 1\}$. Up to isomorphism, any other 2-valued logic may then be seen as a sublogic or fragment of Classical Logic. In 1941, Emil Post studied the lattice of all the 2-valued clones ordered under inclusion [10]. In [11], Wolfgang Rautenberg explored this lattice in order to show that all fragments of Classical Logic are strongly finitely axiomatizable. Rautenberg used an unusual notation and overloaded it several times, causing confusion; in addition, he presented incomplete proofs and made lots of typographical errors, imprecisions and mistakes. In particular, the main fragments of Classical Logic — expression here that refers to those fragments related to the proofs presented by Rautenberg in the first part of his paper — deserve a more rigorous and accessible presentation, because they promote important discussions and results about the remaining fragments. Also, they give bases to the recursive procedures in the second part of the proof of the axiomatizability of all 2-valued fragments. This work proposes a rephrasing of the proofs for the main fragments, with a more modern notation, with more attention to the details and the writing, and with the inclusion of all axiomatizations of the clones under investigation. In addition, the involved proof systems will be specified in the language of the Lean theorem prover, and the derivations necessary for the completeness proofs will be verified with the aid of this tool. In this way, the presentation of the proof of the result given by Rautenberg will be more accessible, understandable and trustworthy to the community.

Keywords: fragments of Classical Logic, Hilbert calculi, Post's lattice, Universal Algebra

List of Figures

List of Hilbert calculi

Contents

1 Introduction

Clones over a set A are sets of operations over A closed under projections and superpositions. From a universal-algebraic consequence-theoretic perspective, Classical Logic can be defined as the logic induced by the complete clone over $\{0, 1\}$. Up to isomorphism, any other 2-valued logic may then be seen as a sublogic or fragment of Classical Logic. For example, the clone generated by the classical conjunction and disjunction induces a proper fragment of Classical Logic.

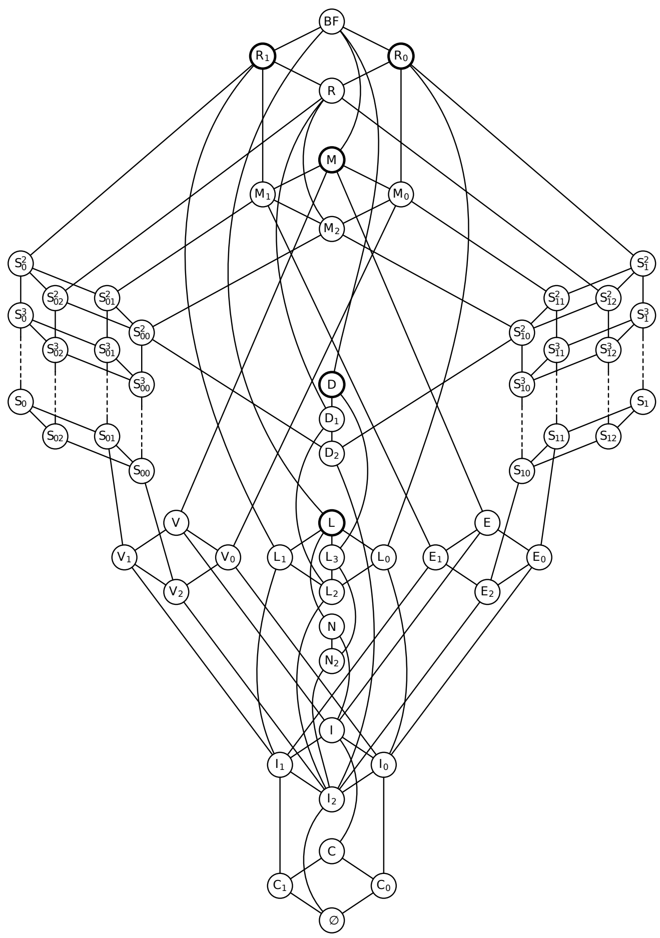

Figure 1: Post's lattice [3].

In 1941, Emil Post studied the lattice of all the 2-valued clones, ordered under inclusion [10]. This lattice (Figure 1) —countably infinite yet constituted of finitely generated members— has constituted ever since an invaluable source of information and insights about the relationships among the sublogics of Classical Logic. Wolfgang Rautenberg, in his study titled "2-element matrices" [11], explored Post's lattice and proved that every 2-valued logic is strongly finitely axiomatizable. This result is specially important for the study of combinations of fragments of Classical Logic [4], since Hilbert calculi constitute more propitious environments for such investigations than other styles of proof systems, like sequent calculi.

The proof of the mentioned result is constructive and is divided into two parts: the main cases, which refer to specific clones located in the finite sections of the lattice; and the reduction procedures, which deal with the infinite portions of the lattice based on the main cases. Despite the relevance and originality of his study, Rautenberg adopted an unusual notation and overloaded it several times for different clones, giving room for confusion and hindering understanding and reproducibility. Moreover, he did not detail various of the proofs of important ancillary results, and his paper contains numerous typographic errors and imprecisions. In addition, some axiomatizations were not fully presented, but had only their existences asserted by the author, and the ones that were presented were done so in ways that do not favour easy and fast reference. Those facts pose obstacles to the study of Rautenberg's work and decrease the confidence in the proof of such an important result for the study of the fragments of Classical Logic under the perspective of the associated Hilbert calculi and their possible combinations. Therefore, a more rigorous and accessible presentation of the proof of this result is necessary.

That being said, the purpose of the present work is to rephrase the proof of the axiomatizability of the thirty-eight clones located in the finite section of Post's lattice, focusing on correctly presenting all related arguments in full level of details and organization, using a more modern and understandable notation, with the support and assurance of the Lean theorem prover. Thus, at the end, we will have a clearly specified and easy to reference calculus for each of those fragments — including those not fully presented by Rautenberg — with its adequacy being justified fundamentally on the basis of formally verified derivations.

Essentially, carrying out this objective amounts to proposing a Hilbert calculus for each of the referred fragments and proving the corresponding soundness and completeness results. Many among such calculi and results were presented by Rautenberg, but need rewriting with a greater level of details, organization, accuracy and in a more clear notation. While the soundness results can be proved using a rather standard procedure, the completeness results need a more sophisticated technique. The one used by Rautenberg and to be also used in this work applies the Lindenbaum-Asser Lemma, considered the most fundamental version of Lindenbaum Lemma to prove completeness in general [2].

Understanding such technique also gives us significant directions to search for rules that lead to complete a calculus with respect to a given fragment of Classical Logic. This is what we will use to produce the axiomatizations that were not fully presented by Rautenberg. The most important case is the clone whose base set is $\{\mathsf{dc}, \mathsf{pt}\}$, where $\mathsf{dc} = \lambda x, y, z.\, (x \land y) \lor (x \land z) \lor (y \land z)$ and $\mathsf{pt} = \lambda x, y, z.\, x \leftrightarrow y \leftrightarrow z$. Although it seems to have one of the most complex calculi among the other clones under consideration, only two of the (interaction) rules were presented by Rautenberg; the other six were found in the present study. Another case is the clone generated by $\{\mathsf{pt}, \neg\}$, for which Rautenberg presented a calculus whose completeness proof was not provided and we could not verify. In order to overcome this, we proposed a modification by adding a new rule of interaction and removing two rules from the calculus given in Rautenberg's paper. In the other hand, we could neither verify the completeness of the calculus proposed by Rautenberg nor propose another one to axiomatize the fragment $\{\mathsf{pt}, \bot\}$. We then chose to keep Rautenberg's suggestion, but highlighting that this case needs further investigation.

The necessary derivations for proving completeness following the aforementioned strategy can be hard to find and to correctly present, because some of them can be very long and involving. This is where Lean comes into play: it will guarantee that each derivation in the course of the completeness proofs is in accordance with the rules of the corresponding proof systems. Lean is a project developed at Microsoft Research that aims to support both mathematical reasoning and reasoning about complex systems, and to verify claims in both domains. It situates automated tools and methods in a framework that supports user interaction and the construction of fully specified axiomatic proofs [1]. Lean also has a very expressive language, a feature to be thoroughly explored here, so much that we will directly use the Lean code to describe derivations in the present document. The Lean code written for the present work is available at `https://github.com/greati/hilbert-classical-fragments` and updates to this document are available at `https://vitorgreati.me/hcclf-monograph`.

We proceed by giving, in Chapter 2, the theoretical background of this work, aiming to fix the concepts and notations to be used in the remaining sections, as well as to give a detailed presentation of Lindenbaum-Asser Lemma and to show how it can be applied for completeness proofs with respect to logical matrices. Then, Chapter 3 focuses on delivering, in a tutorial-like fashion, how we can specify a Hilbert calculus in Lean and prove that a given rule is amongst its derivable rules. In the sequel, Chapter 4 contains the axiomatization of each main fragment of Classical Logic, together with the corresponding adequacy proofs. Finally, Chapter 5 contains the final remarks and suggests future directions of investigation.

2 Theoretical background

2.1 Algebras and clones

A **signature** $\Sigma = \{\Sigma^{(k)}\}_{k \in \omega}$ is a family of sets of function symbols, where each $\# \in \Sigma^{(k)}$ is said to have arity k or, equivalently, to be k-ary. In particular, we use the adjectives "nullary" (or "constant"), "unary", "binary" and "ternary" in reference to symbols with arity zero, one, two and three, respectively. Besides, for convenience, we shall use $\bigcup \Sigma$ in place of Σ when the context removes any risk of ambiguity about a signature Σ.

An **algebra A over a signature** Σ is a structure $\langle A, \cdot^{\mathbf{A}} \rangle$ where $A \neq \varnothing$ is a set dubbed **carrier** or universe of the algebra and each symbol $\# \in \Sigma^{(k)}$ is interpreted as a k-ary operation $\#^{\mathbf{A}}$ over A. Notice that nullary symbols are interpreted as elements of the carrier. When A is finite, we commonly specify the interpretations in $\mathbf{A}$ by tables and, under the set-theoretical perspective that each m-ary $\#^{\mathbf{A}}$ is an $(m+1)$-ary relation over A, we call each of its tuples — the rows of the tables — a **determinant** of $\#^{\mathbf{A}}$.

Example 2.1.1. Consider the signature given by $\Sigma^{(2)} = \{\sqcup, \sqcap\}$ and $\Sigma^{(n)} = \varnothing$, for all $n \neq 2$. An example of an algebra over Σ is $\mathbf{C} := \langle \{\clubsuit, \heartsuit\}, \cdot^{\mathbf{C}} \rangle$, such that

x	y	$\sqcup^{\mathbf{C}}(x,y)$	x	y	$\sqcap^{\mathbf{C}}(x,y)$
$\heartsuit$	$\heartsuit$	$\heartsuit$	$\heartsuit$	$\heartsuit$	$\heartsuit$
$\heartsuit$	$\clubsuit$	$\clubsuit$	$\heartsuit$	$\clubsuit$	$\heartsuit$
$\clubsuit$	$\heartsuit$	$\clubsuit$	$\clubsuit$	$\heartsuit$	$\heartsuit$
$\clubsuit$	$\clubsuit$	$\clubsuit$	$\clubsuit$	$\clubsuit$	$\clubsuit$

By inspecting the table of $\sqcup^{\mathbf{C}}$, we extract the set of its determinants:

$$\{\langle \heartsuit, \heartsuit, \heartsuit \rangle, \langle \heartsuit, \clubsuit, \clubsuit \rangle, \langle \clubsuit, \heartsuit, \clubsuit \rangle, \langle \clubsuit, \clubsuit, \clubsuit \rangle\}.$$

A **homomorphism** between two algebras **A** and **B** over a common signature Σ is a mapping $h : A \to B$ such that $h(\#^{\mathbf{A}}(x_1, \ldots, x_k)) = \#^{\mathbf{B}}(h(x_1), \ldots, h(x_k))$, for each $\# \in \Sigma^{(k)}$ and all $x_1, \ldots, x_k \in A$. An algebra $\mathbf{A}$ is an **absolutely free** algebra **freely generated** by a set $G \subseteq A$ when $\mathbf{A}$ is generated by G (the least algebra over Σ that includes G in its carrier), and any mapping from G to the carrier B of any algebra $\mathbf{B}$ over Σ can be (uniquely) extended to a homomorphism from $\mathbf{A}$ to $\mathbf{B}$.

An important universal-algebraic definition that we will use to define Classical Logic and its fragments is that of a clone of an algebra. We begin by first establishing what is a clone over a set. A **clone** over a set A is a collection $\mathcal{C}$ of operations over A such that

- $\mathcal{C}$ is closed under projections; i.e. every operation $\pi_i^n(x_1, \ldots, x_n) = x_i$, for all $1 \leq i \leq n$ and all $n > 1$, is in $\mathcal{C}$;
- $\mathcal{C}$ is closed under superpositions, that is, whenever an n-ary operation f is in $\mathcal{C}$ and the m-ary operations $g_1, \ldots, g_n$ are in $\mathcal{C}$, the m-ary operation h such that $h(x_1, \ldots, x_m) = f(g_1(x_1, \ldots, x_m), \ldots, g_n(x_1, \ldots, x_m))$ is also in $\mathcal{C}$. We say here that the operation h obtained this way is **f-headed**; and
- whenever a constant function f is in $\mathcal{C}$, i.e. $f(x_1, \ldots, x_n) = a$ for all $x_1, \ldots, x_n \in A$ and a fixed $a \in A$, the nullary operation $f_a = a$ is also in $\mathcal{C}$.

Clearly, the set of all operations (of every possible arity) over A is a clone, called the **complete clone over** A and denoted by $\mathcal{C}^A$.

Given a set $\mathcal{F}$ of operations over A, the **clone generated by** $\mathcal{F}$, denoted here by $\mathsf{Clo}(\mathcal{F})$, is the smallest clone that contains $\mathcal{F}$. The **clone of an algebra $\mathbf{M}$** over Σ, namely $\mathsf{Clo}(\mathbf{M})$, is the clone generated by the set of the fundamental operations of $\mathbf{M}$, which are those that interpret the symbols in Σ (check more results on this topic in [9, 8]).

We close this section with the notion of monotonic operations, important to characterize some fragments and produce their axiomatizations (see Section 4.6). An m-ary operation f over $\{0, 1\}$ (seen here as a totally ordered set) is **monotonic** when, for all $\vec{x} = \langle x_1, \ldots, x_m \rangle$, $\vec{y} = \langle y_1, \ldots, y_m \rangle \in \{0, 1\}^m$, if $\vec{x} \leq \vec{y}$, then $f(\vec{x}) \leq f(\vec{y})$, where $\leq$ is the lexicographical order on the cartesian product $\{0, 1\}^m$.

Example 2.1.2. The set of all monotonic operations over $\{0, 1\}$ is a clone (see Section 4.14).

2.2 Languages and logics

A **language** over a signature Σ generated by a countable set $P = \{\mathrm{p}_i \mid i \in \omega\}$ of **propositional variables**, denoted by $\mathbf{L}_\Sigma^P$, is the absolutely free algebra over Σ freely generated by P, where each k-ary symbol of Σ is assigned to some k-ary operation (commonly referred to as **connective** in this context) on L_Σ^P (the carrier of the language), denoted here by the same symbol being interpreted. When the set P is clear from the context, we shall omit it from the notation.

The countably-many elements of a language are called **formulas**, denoted here by capital Roman letters (A, B, ...). Propositional variables, together with nullary connectives, are called **atomic formulas**. A formula resulting from the application of a connective of arity greater than one is called a **compound**, and the total number of connectives involved in its construction is its **complexity**. When A is a compound formula $\#(\mathrm{A}_1, \ldots, \mathrm{A}_n)$, we say that A is **#-headed**. Moreover, where A is a formula, by $\mathsf{Vars}(\mathrm{A})$ we will denote the set of propositional variables occurring in A.

The conception of language just introduced gives us two important properties: each formula in a language is built up from atomic formulas by the application of its fundamental connectives in only one way (unique readability); and, for any algebra **B** over the same signature as the language, any function from the set P of propositional variables into B can be uniquely extended to a homomorphism from the language to **B**. An equivalent formulation of the latter is to say that every such homomorphism is uniquely determined by its restriction to the set of propositional variables.

Example 2.2.1. A conventional signature Σ_{CL} for Classical Logic is such that ${\Sigma_{CL}}^{(0)} = \{\bot, \top\}$, ${\Sigma_{CL}}^{(1)} = \{\neg\}$, ${\Sigma_{CL}}^{(2)} = \{\vee, \wedge, \to\}$ and ${\Sigma_{CL}}^{(n)} = \varnothing$, for all $n > 2$. Examples of formulas of $\mathbf{L}_{\Sigma_{CL}}^P$ are p_{12}, $\neg\mathrm{p}_1$, $\neg(\mathrm{p}_2 \vee \mathrm{p}_5)$, $\mathrm{p}_2 \to \bot$, $(\mathrm{p}_1 \wedge \mathrm{p}_2) \vee (\mathrm{p}_5 \to \mathrm{p}_6)$ and $\top$.

We can also consider a formula $\mathrm{A} \in L_\Sigma^{\{\mathrm{p}_{i_1},\ldots,\mathrm{p}_{i_m}\}}$, with $i_l \in \omega$ and $1 \leq l \leq m$, as an operation $\lambda\mathrm{p}_{i_1}, \ldots, \mathrm{p}_{i_m}.\mathrm{A}$, said to be an m-ary **derived connective** of $\mathbf{L}_\Sigma^P$.

Whenever Σ and Σ' are signatures such that $\Sigma \subseteq \Sigma'$, we say that Σ is a **fragment** of Σ'. Equivalently, Σ' is said to be an **expansion** of Σ. The same terminology may be applied to $\mathbf{L}_\Sigma^P$ and $\mathbf{L}_{\Sigma'}^P$, since $\Sigma \subseteq \Sigma'$ implies $L_\Sigma^P \subseteq L_{\Sigma'}^P$.

Example 2.2.2. Let $\Sigma_\to$ be such that ${\Sigma_\to}^{(2)} = \{\to\}$ and ${\Sigma_\to}^{(n)} = \varnothing$, for all $n \neq 2$. Then $\Sigma_\to$ is a fragment (the implicational fragment) of Σ_{CL}.

When we deal with expansions of a language by new connectives, sometimes it is necessary to have a representation of the formulas of the expansion in terms of the connectives of the original one, assigning atomic representations to formulas involving the new connectives. In order to formalize such representation, let Σ and Σ^+ be two signatures, such that $\Sigma \subseteq \Sigma^+$. Also, let P, V and V^+ be denumerable sets of propositional variables, where V and V^+ are disjoint, and consider bijections $f : P \to V$ and $f^+ : L^P_{\Sigma^+ \setminus \Sigma} \to V^+$. Then, we define the Σ-**skeleton** of a formula in $L^P_{\Sigma^+}$ as the result of applying this formula to the function $\mathsf{sk}^{f,f^+}_\Sigma : L^P_{\Sigma^+} \to L^{V \cup V^+}_\Sigma$, such that:

$$\mathsf{sk}^{f,f^+}_\Sigma(\mathrm{C}) = \begin{cases} f(\mathrm{C}) & \mathrm{C} \in P, \\ \#(\mathsf{sk}^{f,f^+}_\Sigma(\mathrm{C}_1), \ldots, \mathsf{sk}^{f,f^+}_\Sigma(\mathrm{C}_n)) & \mathrm{C} = \#(\mathrm{C}_1, \ldots, \mathrm{C}_n) \text{ and } \# \in \Sigma^{(n)}, \\ f^+(\mathrm{C}) & \text{otherwise.} \end{cases}$$

Notice that the function just defined is a bijection and its inverse has a very similar definition, essentially changing the application of f and f^+ by their respective inverses.

Example 2.2.3. Take $\Sigma^{(2)} = \{\wedge\}$, $\Sigma^{+(2)} = \{\wedge, \vee\}$, and $\Sigma^{(n)} = \Sigma^{+(n)} = \varnothing$ for all $n \neq 2$. Also, let $V = \{\mathrm{p}_{2i} \in P \mid i \in \omega\}$, $V^+ = \{\mathrm{p}_{2i+1} \in P \mid i \in \omega\}$, $f(\mathrm{p}_i) = \mathrm{p}_{2i}$, and $f^+(\mathrm{A}_i) = \mathrm{p}_{2i+1}$, for a fixed enumeration $\{\mathrm{A}_i\}_{i \in \omega}$ of the formulas in $L^P_{\Sigma^+ \setminus \Sigma}$. Then, assuming that $\mathrm{C} := (\mathrm{p}_2 \vee \mathrm{p}_3) \wedge \mathrm{p}_4$ has index 2 in the enumeration, we have $\mathsf{sk}^{f,f^+}_\Sigma(\mathrm{C}) = \mathrm{p}_5 \wedge \mathrm{p}_8$.

We define now a mechanism for representing a connective in a signature in terms of derived connectives in a different language. Given two signatures Ξ and Σ, a (homophonic) **translation** is a mapping $\mathbf{t} : \Xi \to L^P_\Sigma$ such that, for each $\# \in \Xi^{(k)}$, $\mathbf{t}(\#) \in L^{\{\mathrm{p}_1,\ldots,\mathrm{p}_k\}}_\Sigma$, with $\mathbf{t}(\#)$ interpreted as a k-ary derived connective of $\mathbf{L}^P_\Sigma$. A translation $\mathbf{t}$ naturally extends to a function $\mathbf{t} : L^P_\Xi \to L^P_\Sigma$ by letting $\mathbf{t}(\mathrm{p}) = \mathrm{p}$, for $\mathrm{p} \in P$, and $\mathbf{t}(\#(\mathrm{A}_1, \ldots, \mathrm{A}_k)) := \mathbf{t}(\#)(\mathbf{t}(\mathrm{A}_1), \ldots, \mathbf{t}(\mathrm{A}_k))$. We will often use translations in this study to express non-conventional connectives of some fragments of Classical Logic in terms of the conventional ones, like $\wedge$, $\vee$ and $\to$.

Given a language $\mathbf{L}^P_\Sigma$, a **substitution** is an endomorphism on $\mathbf{L}^P_\Sigma$. The collection of all substitutions over the language is denoted by $\mathsf{Sb}(\mathbf{L}^P_\Sigma)$. The application of a substitution σ to a formula A is denoted by $\sigma(\mathrm{A})$ or A^σ and this naturally extends to each set of formulas Π by letting $\sigma(\Pi) = \Pi^\sigma = \{\mathrm{A}^\sigma \mid \mathrm{A} \in \Pi\}$.

A **consequence relation** over the language $\mathbf{L}^P_\Sigma$ is a relation $\vdash \subseteq \mathsf{Pow}(L^P_\Sigma) \times L^P_\Sigma$ respecting, for every $\Gamma \cup \Delta \cup \{\mathrm{A}\} \subseteq L^P_\Sigma$, the following properties (read Γ, Δ as $\Gamma \cup \Delta$):

(R) Reflexivity: $\mathrm{A} \vdash \mathrm{A}$;

(M) Monotonicity: if $\Gamma \vdash \mathrm{A}$, then $\Gamma, \Delta \vdash \mathrm{A}$;

(T) Transitivity: if $\Gamma \vdash \mathrm{B}$ for every $\mathrm{B} \in \Delta$ and $\Gamma, \Delta \vdash \mathrm{A}$, then $\Gamma \vdash \mathrm{A}$; and

(S) Substitution-invariance: if $\Gamma \vdash \mathrm{A}$, then $\Gamma^\sigma \vdash \mathrm{A}^\sigma$.

We will often use in this work, under the same name, the specialized version of (T) in which Δ is a singleton.

A consequence relation is **finitary** when it respects

(F) Finitariness: if $\Gamma \vdash \mathrm{A}$, then there is a finite set $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash \mathrm{A}$.

A **logic** $\mathcal{L}$ over a signature Σ is then a structure $\langle \Sigma, \vdash \rangle$, where $\vdash$ is a consequence relation over L^P_Σ. We shall sometimes talk about a logic by referring directly to its consequence relation. Assertions of the form $\Gamma \vdash \mathrm{A}$ are called **consecutions**, and may be read as "A follows from Γ (according to $\mathcal{L}$)". A formula A for which $\vdash \mathrm{A}$ is said to be a **theorem** of $\mathcal{L}$. A set $\Gamma \subseteq L^P_\Sigma$ is a **theory** of $\mathcal{L}$ whenever $\Gamma \vdash \mathrm{A}$ implies $\mathrm{A} \in \Gamma$.

A set $\Gamma \subseteq L$ is **maximal** (consistent) with respect to a consequence relation $\vdash$ over the language $\mathbf{L}$ when $\Gamma \nvdash \mathrm{A}$ for some $\mathrm{A} \in L$ and, for any $\mathrm{A} \notin \Gamma$, we have $\Gamma, \mathrm{A} \vdash \mathrm{B}$ for all $\mathrm{B} \in L$. The set Γ is **relatively maximal** with respect to $\vdash$ when there is a formula $\mathrm{Z} \in L$ such that $\Gamma \nvdash \mathrm{Z}$ and $\Gamma, \mathrm{A} \vdash \mathrm{Z}$ for any $\mathrm{A} \notin \Gamma$. Under those conditions, we also say that such Γ is Z-**maximal** with respect to $\vdash$.

We present now the notion of axiomatic expansion, which eases the axiomatization of numerous fragments of Classical Logic. Let Σ and Σ^+ be two signatures, such that $\Sigma \subseteq \Sigma^+$, that is, Σ^+ expands Σ. Also, let $\vdash$ and $\vdash^+$ be consequence relations over $\mathbf{L}_\Sigma$ and $\mathbf{L}_{\Sigma^+}$ respectively. Then, whenever $\vdash \subseteq \vdash^+$, $\vdash^+$ is said to be an **expansion** of $\vdash$. Given a set $\Lambda \subseteq L_{\Sigma^+}$, the **axiomatic expansion of $\vdash$ induced by** Λ, denoted by $\vdash_{+\Lambda}$, is the least expansion of $\vdash$ where $\vdash_{+\Lambda} \mathrm{A}^\sigma$, for each $\mathrm{A} \in \Lambda$ and $\sigma \in \mathsf{Sb}(\mathbf{L}^P_\Sigma)$.

Some important properties of a logic may be described as rules over linguistic objects called sequents. A **sequent** over a signature Σ is an object of the form $\Gamma \succ \mathrm{A}$, where $\Gamma \cup \{\mathrm{A}\} \subseteq L^P_\Sigma$ is finite. An n-ary **sequent-style rule** r is given by a sequence of sequents $\Delta_1 \succ \mathrm{A}_1, \ldots, \Delta_n \succ \mathrm{A}_n, \Delta \succ \mathrm{A}$, where the last is the conclusion sequent and the others are the premiss sequents. Another notation for r is

$$\frac{\Delta_1 \succ \mathrm{A}_1 \quad \cdots \quad \Delta_n \succ \mathrm{A}_n}{\Delta \succ \mathrm{A}} \; \mathsf{r}$$

We say that such sequent-style rule **holds** in a consequence relation $\vdash$ over Σ when, for all $\sigma \in \mathsf{Sb}(\mathbf{L}_\Sigma^P)$ and all $\Gamma \subseteq L_\Sigma^P$,

$$\text{if } \Gamma, \Delta_1^\sigma \vdash \mathrm{A}_1^\sigma \text{ and } \ldots \text{ and } \Gamma, \Delta_n^\sigma \vdash \mathrm{A}_n^\sigma \text{ then } \Gamma, \Delta^\sigma \vdash \mathrm{A}^\sigma.$$

Example 2.2.4. The following rules hold in any consequence relation, as a direct consequence of properties (R), (M) and (T):

$$\frac{}{\mathrm{A} \succ \mathrm{A}}\,\mathsf{R} \qquad \frac{\mathrm{A} \succ \mathrm{B}}{\mathrm{C}, \mathrm{A} \succ \mathrm{B}}\,\mathsf{M}_1 \qquad \frac{\mathrm{A} \succ \mathrm{B} \quad \mathrm{B} \succ \mathrm{C}}{\mathrm{A} \succ \mathrm{C}}\,\mathsf{T}_1 \qquad \frac{\mathrm{D}, \mathrm{A} \succ \mathrm{B} \quad \mathrm{D}, \mathrm{B} \succ \mathrm{C}}{\mathrm{D}, \mathrm{A} \succ \mathrm{C}}\,\mathsf{T}_2$$

Sequent-style rules will be specially important because the properties that lead to the completeness of the calculi discussed in this work can be seen as sequent-style rules and, as we will see in Section 2.8, they are preserved in axiomatic expansions, a result that automatically provides axiomatizations for expansions by the constant $\top$.

Finally, we say that a function $g : L_\Sigma^P \to \{0, 1\}$ is **consistent with** a consequence relation $\vdash$ over Σ when, for all $\Gamma \cup \{\mathrm{A}\}$ such that $\Gamma \vdash \mathrm{A}$, if $g(\Gamma) \subseteq \{1\}$, then $g(\mathrm{A}) = 1$.

2.3 Hilbert calculi

A **Hilbert calculus** $\mathscr{H}$ is a structure $\langle \Sigma, R \rangle$, where Σ is a signature and R is the set of **inference rules**, where each $\mathsf{r} \in R$ is a relation $\mathsf{r} \subseteq {L_\Sigma}^{n+1}$, with $n \in \omega$ being its arity. Nullary rules are dubbed **axioms**. Moreover, the way we specify an n-ary rule in this work is by a schema of the form

$$\frac{\mathrm{A}_1 \quad \ldots \quad \mathrm{A}_n}{\mathrm{A}_{n+1}}\,\mathsf{r}\;.$$

By this we mean that $\langle {\mathrm{A}_1}^\sigma, \ldots, {\mathrm{A}_n}^\sigma, {\mathrm{A}_{n+1}}^\sigma \rangle \in \mathsf{r}$ for every substitution σ. We may write such schema in inline form using the syntax $(\mathsf{r})\ \mathrm{A}_1, \ldots, \mathrm{A}_n/\mathrm{A}_{n+1}$. The formulas $\mathrm{A}_1, \ldots, \mathrm{A}_n$ are called **premisses**, while A_{n+1} is the **conclusion** of r. Each $\rho \in \mathsf{r}$ is called an **instance** of r. A rule involving more than one connective is called an **interaction rule** or mixing rule.

A Hilbert calculus $\mathscr{H} := \langle \Sigma, R \rangle$ induces a logic $\mathcal{L}_\mathscr{H} := \langle \Sigma, \vdash_\mathscr{H} \rangle$, where $\Gamma \vdash_\mathscr{H} \mathrm{B}$ if there is a sequence of formulas $\mathrm{B}_1, \ldots, \mathrm{B}_k$, with $k > 1$, such that, for each $1 \leq i \leq k$, either (i) $\mathrm{B}_i \in \Gamma$, or (ii) B_i is an instance of an axiom of $\mathscr{H}$, or (iii) B_i results from an application of an m-ary rule $\mathsf{r} \in R$ to earlier formulas $\mathrm{B}_{i_1}, \ldots, \mathrm{B}_{i_m}$ in the sequence, i.e. $\langle \mathrm{B}_{i_1}, \ldots, \mathrm{B}_{i_m}, \mathrm{B}_i \rangle \in \mathsf{r}$, with $i_l < i$ for all $1 \leq l \leq m$. The aforementioned sequence is called a **derivation** or deduction of B from Γ. Aside from having properties (R), (M) and

(T), the relation $\vdash_{\mathscr{H}}$ is also finitary given the finite character of the derivations. When $C_1^\sigma, \ldots, C_n^\sigma \vdash_{\mathscr{H}} C_{n+1}^\sigma$, for some $C_i \in L_\Sigma^P$, with $1 \leq i \leq n$, and all $\sigma \in \mathsf{Sb}(\mathbf{L}_\Sigma^P)$, we say that the rule

$$\frac{C_1 \quad \ldots \quad C_n}{C_{n+1}}$$

is **derivable** in $\mathscr{H}$.

Example 2.3.1. A proof-theoretical characterization of Classical Logic is given by the following Hilbert Calculus over Σ_{CL}:

Hilbert Calculus 1. $\mathscr{B}$

$$\frac{A \quad A \to B}{B}\ \mathsf{cl}_1$$

$$\frac{}{A \to (B \to A)}\ \mathsf{cl}_2$$

$$\frac{}{(A \to (B \to C)) \to ((A \to B) \to (A \to C))}\ \mathsf{cl}_3$$

$$\frac{}{(A \wedge B) \to A}\ \mathsf{cl}_4$$

$$\frac{}{(A \wedge B) \to B}\ \mathsf{cl}_5$$

$$\frac{}{A \to (B \to (A \wedge B))}\ \mathsf{cl}_6$$

$$\frac{}{A \to (A \vee B)}\ \mathsf{cl}_7$$

$$\frac{}{B \to (A \vee B)}\ \mathsf{cl}_8$$

$$\frac{}{(A \to C) \to ((B \to C) \to ((A \vee B) \to C))}\ \mathsf{cl}_9$$

$$\frac{}{(A \to B) \to ((A \to \neg B) \to \neg A)}\ \mathsf{cl}_{10}$$

$$\frac{}{\neg\neg A \to A}\ \mathsf{cl}_{11}$$

$$\frac{}{\top}\ \mathsf{cl}_{12}$$

$$\frac{}{\bot \to \mathrm{A}}\ \mathsf{cl}_{13}$$

2.4 Logical matrices

A **logical matrix** $\mathbb{M}$ over a signature Σ is a structure $\langle \mathbf{M}, D\rangle$, where $\mathbf{M}$ is an algebra over Σ whose carrier M is the collection of **truth-values**, and $D \subseteq M$ is the set of **designated** values. For each $\# \in \Sigma$, let $\#^{\mathbb{M}}$ denote the interpretation of $\#$ in the algebra $\mathbf{M}$, that is $\#^{\mathbb{M}} := \#^{\mathbf{M}}$. The definition of a clone naturally extends to the one of the **clone of a logical matrix**, by letting $\mathsf{Clo}(\mathbb{M}) := \mathsf{Clo}(\mathbf{M})$.

A **valuation over** $\mathbb{M}$ or an $\mathbb{M}$-valuation is a homomorphism from the algebra language $\mathbf{L}^P_\Sigma$ into $\mathbf{M}$; in other words, the set of all such valuations, denoted here by $\mathsf{Val}_P(\mathbb{M})$, coincides with $\mathsf{Hom}(\mathbf{L}^P_\Sigma, \mathbf{M})$. We say that a valuation v over $\mathbb{M}$ satisfies a formula A if $v(\mathrm{A}) \in D$ and that it satisfies a set of formulas Γ if $v(\Gamma) \subseteq D$, with $v(\Gamma) = \{v(\mathrm{A}) \mid \mathrm{A} \in \Gamma\}$. Where $\Gamma \cup \{\mathrm{A}\} \subseteq L^P_\Sigma$, let $\Gamma \vdash_{\mathbb{M}} \mathrm{A}$ if and only if every $\mathbb{M}$-valuation that satisfies Γ also satisfies A. Then we can show that $\mathcal{L}_{\mathbb{M}} = \langle \Sigma, \vdash_{\mathbb{M}}\rangle$ is a logic (i.e. $\vdash_{\mathbb{M}}$ satisfies (R), (M), (T) and (S)) and we call it the **logic characterized by** $\mathbb{M}$.

A Hilbert calculus $\mathscr{H}$ is **sound** with respect to a logical matrix $\mathbb{M}$ when $\vdash_{\mathscr{H}} \subseteq \vdash_{\mathbb{M}}$. Conversely, it is **complete** with respect to $\mathbb{M}$ when $\vdash_{\mathbb{M}} \subseteq \vdash_{\mathscr{H}}$. In addition, the calculus $\mathscr{H}$ **axiomatizes** $\mathbb{M}$ when it is sound and complete with respect to $\mathbb{M}$. We also say that this calculus is an axiomatization or is adequate for $\mathbb{M}$.

Finally, we present a key construction that allows us to produce a matrix for a given signature based on a known matrix and a translation. Given signatures Ξ and Σ, a translation $\mathbf{t} : \Xi \to L^P_\Sigma$ and a logical matrix $\mathbb{M} = \langle \mathbf{M}, D\rangle$ over Σ, we may say that $\mathbb{M}$ induces an interpretation $\#^{\mathbb{M}} : M^k \to M$ under $\mathbf{t}$ for each $\# \in \Xi^{(k)}$, such that $\#^{\mathbb{M}}(a_1, \ldots, a_k) = v(\mathbf{t}(\#))$, where v is any $\mathbb{M}$-valuation for which $v(\mathrm{p}_i) = a_i$ for $1 \leq i \leq k$. We denote by $\mathbb{M}^{\mathbf{t}}$ the logical matrix over Ξ with the same truth-values and designated values as $\mathbb{M}$ whose interpretations for each symbol in Ξ are the induced interpretations under $\mathbf{t}$.

2.5 Classical Logic and its fragments

For any desired signature Σ, denote by $\mathbf{2}_\Sigma$ an algebra over Σ whose carrier is the set $\{0, 1\}$ and by $\mathbb{2}_\Sigma$ the logical matrix $\langle \mathbf{2}_\Sigma, \{1\}\rangle$, also called here a **2-matrix**. Additionally, let $\mathcal{B}_\Sigma$ be the logic characterized by $\mathbb{2}_\Sigma$, i.e. $\mathcal{B}_\Sigma := \mathcal{L}_{\mathbb{2}_\Sigma}$. **Classical logic** is then defined as the logic characterized by any 2-matrix $\mathbb{2}_\Sigma$ such that $\mathsf{Clo}(\mathbb{2}_\Sigma) = \mathcal{C}^{\{0,1\}}$.

Example 2.5.1. A common matrix for Classical Logic is $\mathbb{2} := \mathbb{2}_{\Sigma_{CL}}$, whose algebra is $\mathbf{2}$, given previously.

It is well-known that some combinations of these operations (for instance, those for $\neg$ and $\wedge$) are functionally complete, so $\mathsf{Clo}(\mathbb{2}_{\Sigma_{CL}}) = \mathcal{C}^{\{0,1\}}$.

Any other 2-matrix characterizes a **fragment of Classical Logic**. Of course, the ones of interest in this work are the **proper** fragments. Henceforth, whenever $\# \in \Sigma_{CL}$ appears in the context of a fragment, we will use the interpretation $\#^{\mathbb{2}}$ given in the previous example. Moreover, whenever connectives other than those of Σ_{CL} appear in a signature Σ of a fragment of Classical Logic, we will present a translation $\mathbf{t} : \Sigma \to L^P_{\Sigma_{CL}}$ such that $\mathbb{2}_\Sigma = \mathbb{2}^{\mathbf{t}}_{\Sigma_{CL}}$.

Example 2.5.2. The matrix $\mathbb{2}_{\wedge,\vee,\top,\bot}$ is a proper fragment of Classical Logic, since antitone functions like $\neg^{\mathbb{2}}$ are not in $\mathsf{Clo}(\mathbb{2}_{\wedge,\vee,\top,\bot})$.

In [10], Emil Post studied the collection of all clones over $\{0, 1\}$, characterizing it as a countable[1] lattice ordered under inclusion, known as **Post's lattice**, whose members are all **finitely generated**, i.e. generated by a finite set of operations over $\{0, 1\}$. This means that there is a corresponding 2-matrix for each of those clones, allowing us to see Post's lattice as the lattice of all fragments of Classical Logic. Since its characterization, this lattice has been used to investigate properties of such fragments and their relationships.

In one of these explorations of Post's lattice, Wolfgang Rautenberg encountered finite axiomatizations for each fragment of Classical Logic [11]. In the next chapter, we will study most of them in detail. As a convention, for any given Σ, we will denote by $\mathscr{B}_\Sigma$ the calculus that we claim to be adequate for the classical fragment $\mathbb{2}_\Sigma$.

2.6 Lindenbaum-Asser Lemma

In this section, we present in details the Lindenbaum-Asser Lemma, which allows us to produce relatively maximal theories for non-trivial finitary logics. This result is a powerful tool for proving the completeness of a calculus and it is used exhaustively throughout this work.

[1]Curiously, collections of clones over sets with larger cardinality turn out to be uncountable.

Theorem 2.6.1. *For any finitary consequence relation* $\vdash \subseteq \mathsf{Pow}(L) \times L$ *and any* $\Gamma \cup \{\mathrm{Z}\} \subseteq L$*, if* $\Gamma \nvdash \mathrm{Z}$*, then there exists a set* $\Gamma^+ \supseteq \Gamma$ *such that*
(i) $\Gamma^+ \nvdash \mathrm{Z}$*;*
(ii) $\Gamma^+, \mathrm{B} \vdash \mathrm{Z}$*, for any* $\mathrm{B} \notin \Gamma^+$*; and*
(iii) the characteristic function of Γ^+ *is consistent with* $\vdash$*, verifying all of* Γ^+ *and falsifying* Z.

Proof. Let $\vdash$ be a finitary consequence relation over a language L. Suppose that $\Gamma \nvdash \mathrm{Z}$, for some $\Gamma \cup \{\mathrm{Z}\} \subseteq L$. Then consider an enumeration $\mathrm{A}_1, \ldots, \mathrm{A}_n, \ldots$ of the formulas of L, and define the sequence $\{\Gamma_i\}_{i=0}^{\infty}$ by setting

$$\Gamma_0 = \Gamma$$

$$\Gamma_{i+1} = \begin{cases} \Gamma_i \cup \{\mathrm{A}_{i+1}\} & \text{if } \Gamma_i, \mathrm{A}_{i+1} \nvdash \mathrm{Z}, \\ \Gamma_i & \text{otherwise.} \end{cases}$$

Finally, define $\Gamma^+ := \bigcup_{i=0}^{\infty} \Gamma_i$. First of all, notice that (a): $\Gamma_i \nvdash \mathrm{Z}$, for each i, a fact that can be easily proved by induction: the base case holds because $\Gamma \nvdash \mathrm{Z}$, and assuming $\Gamma_k \nvdash \mathrm{Z}$, for some $k > 0$, directly from the above construction and the induction hypothesis we get $\Gamma_{k+1} \nvdash \mathrm{Z}$. Since $\Gamma = \Gamma_0 \subseteq \Gamma^+$, it remains to verify (i), (ii) and (iii).

For (i), suppose that $\Gamma^+ \vdash \mathrm{Z}$. Then, since $\vdash$ is finitary, there is some finite $\Theta \subseteq \Gamma^+$ such that $\Theta \vdash \mathrm{Z}$. Since $\Theta \subseteq \Gamma_i$, for some i, we get, by (M), $\Gamma_i \vdash \mathrm{Z}$, which contradicts (a). Then $\Gamma^+ \nvdash \mathrm{Z}$.

For (ii), if $\mathrm{B} \notin \Gamma^+$, then $\mathrm{B} \notin \Gamma$ and, in case $\mathrm{B} = \mathrm{A}_{k+1}$, $\Gamma_{k+1} = \Gamma_k$ (otherwise $\mathrm{A}_{k+1} = \mathrm{B} \in \Gamma^+$), what is possible only if $\Gamma, \mathrm{A}_{k+1} \vdash \mathrm{Z}$, according to the construction given above.

For (iii), let μ^+ be the characteristic function of Γ^+, that is $\mu^+(\mathrm{A}) = 1$ if, and only if, $\mathrm{A} \in \Gamma^+$. Clearly, $\mu^+(\Gamma) \subseteq \{1\}$. By assuming $\mu^+(\mathrm{Z}) = 1$, we have $\mathrm{Z} \in \Gamma^+$, and, by appealing to (R) and (M), $\Gamma^+ \vdash \mathrm{Z}$, contradicting (i). Consistency with $\vdash$ can be proved again by contradiction: assume that μ^+ is not consistent with $\vdash$. Then, there is $\Theta \cup \{\mathrm{A}\} \subseteq L$ such that $\Theta \vdash \mathrm{A}$, but $\mu^+(\Theta) \subseteq \{1\}$ and $\mu^+(\mathrm{A}) = 0$. Because $\vdash$ is finitary, $\Theta_0 \vdash \mathrm{A}$ for some finite $\Theta_0 \subseteq \Theta$. Because μ^+ is the characteristic function of Γ^+, $\mathrm{A} \notin \Gamma^+$ and, using that $\mu^+(\Theta_0) \subseteq \{1\}$, $\Theta_0 \subseteq \Gamma^+$. Let k be the greatest index of the formulas in Θ_0 and l be the index of A, considering the enumeration $\mathrm{A}_1, \ldots, \mathrm{A}_n, \ldots$ used in the construction of Γ^+. Thus $\Theta_0 \subseteq \Gamma_k$, and, because $\Theta_0 \vdash \mathrm{A}$, (b): $\Gamma_k \vdash \mathrm{A}$ by (M). Moreover, since $\mathrm{A} \notin \Gamma^+$, $\mathrm{A} \notin \Gamma_l$, meaning that (c): $\Gamma_l, \mathrm{A} \vdash \mathrm{Z}$, by (ii). Let $m = \max\{k, l\}$, then $\Gamma_k, \Gamma_l \subseteq \Gamma_m$. This fact, together with (b) and (c), leads to (b)': $\Gamma_m \vdash \mathrm{A}$ and (c)': $\Gamma_m, \mathrm{A} \vdash \mathrm{Z}$. By (T) from (b)' and (c)', $\Gamma_m \vdash \mathrm{Z}$, which, by (M), implies $\Gamma^+ \vdash \mathrm{Z}$, contradicting (*i*). □

Corollary 2.6.1.1. *For* Γ^+ *and* Z *as given in Theorem 2.6.1, we have* $\Gamma^+, \mathrm{B} \nvdash \mathrm{Z}$ iff $\mathrm{B} \in \Gamma^+$.

Proof. The left-to-right direction is the contrapositive version of (*ii*). For the right-to-left, the proof goes by contradiction: suppose that $\Gamma^+, \mathrm{B} \vdash \mathrm{Z}$ and $\mathrm{B} \in \Gamma^+$. Then $\Gamma^+ \vdash \mathrm{Z}$, which is impossible in view of Theorem 2.6.1 (*i*). □

Corollary 2.6.1.2. *The relatively maximal set* Γ^+ *constructed in the proof of Theorem 2.6.1 is deductively closed, namely, for all* $\mathrm{A} \in L$,

$$\Gamma^+ \vdash \mathrm{A} \text{ iff } \mathrm{A} \in \Gamma^+.$$

Proof. Suppose that Γ^+ is Z-maximal. The right-to-left direction follows by appealing to (R) and (M). For the left-to-right direction, using the contrapositive version, assume that (a): $\mathrm{A} \notin \Gamma^+$. The proof goes by contradiction: suppose that (b): $\Gamma^+ \vdash \mathrm{A}$. Because of (a), Theorem 2.6.1 (*ii*) leads to (c): $\Gamma^+, \mathrm{A} \vdash \mathrm{Z}$. Then, (T) applied to (b) and (c) results in $\Gamma^+ \vdash \mathrm{Z}$, contradicting the fact that Γ^+ is Z-maximal. □

Finally, the following result gives sufficient conditions for a relatively maximal set being nonempty when we are dealing with consequence relations sound with respect to some logical matrix. It plays an important role in some completeness proofs presented in Chapter 4.

Lemma 2.6.2. *If* $\vdash$ *is a consequence relation over* Σ *and* $\vdash \subseteq \vdash_{\mathbb{M}}$, *for some logical matrix* $\mathbb{M}$ *having no tautologies and at least one designated value, then every* Z*-maximal set with respect to* $\vdash$ *is non-empty.*

Proof. Let $\Gamma^+ \subseteq L_\Sigma^P$ be a Z-maximal set with respect to some consequence relation $\vdash$ over Σ, and suppose that $\vdash \subseteq \vdash_{\mathbb{M}}$ for some logical matrix $\mathbb{M}$ having no tautologies and at least one designated value, say a. We want to show that $\Gamma^+ \neq \varnothing$. Work by contradiction: suppose that $\Gamma^+ = \varnothing$, then $\mathrm{A} \vdash \mathrm{Z}$ for each formula A, by Corollary 2.6.1.1. In particular, (a): $\mathrm{p} \vdash \mathrm{Z}$, for some propositional variable $\mathrm{p} \notin \mathsf{Vars}(\mathrm{Z})$. Since Z is at least a contingent formula with respect to $\mathbb{M}$, take an $\mathbb{M}$-valuation v such that $v(\mathrm{Z}) = u$, for some undesignated value u, and consider another valuation $v^\star$ that agrees with v but gives the assignment $v^\star(\mathrm{p}) = a$. Then, $v^\star(\mathrm{p}) = a$, but $v^\star(\mathrm{Z}) = u$, hence $\mathrm{p} \nvdash_{\mathbb{M}} \mathrm{Z}$ and so $\mathrm{p} \nvdash \mathrm{Z}$, contradicting (a). □

2.7 Completeness via relatively maximal theories

We describe now a general procedure for proving the completeness result of a calculus with respect to a fragment of Classical Logic. Given a signature Σ, a logical matrix 2_Σ and a Hilbert calculus $\mathscr{B}$ over Σ, the (strong) completeness result consists in proving that $\vdash_{2_\Sigma} \subseteq \vdash_{\mathscr{B}}$. By contraposition, this amounts to showing that, for all $\Gamma \subseteq L_\Sigma$ and $A \in L_\Sigma$,

$$\text{if} \quad \Gamma \nvdash_{\mathscr{B}} A \quad \text{then} \quad \Gamma \nvdash_{2_\Sigma} A.$$

With this purpose, assume that $\Gamma \nvdash_{\mathscr{B}} Z$, for some $\Gamma \subseteq L_\Sigma$ and $Z \in L_\Sigma$. Since $\vdash_{\mathscr{B}}$ is finitary, the Lindenbaum-Asser Lemma (Section 2.6) guarantees that there is a set $\Gamma^+ \supseteq \Gamma$ such that $\Gamma^+ \nvdash_{\mathscr{B}} Z$ and the characteristic function μ^+ of Γ^+ is a candidate for a 2_Σ-valuation consistent with $\vdash_{\mathscr{B}}$. Since $\Gamma \subseteq \Gamma^+$, $\mu^+(\Gamma) \subseteq \{1\}$ also holds, meaning that $\Gamma \nvdash_{2_\Sigma} Z$, the desired result for completeness.

The crucial step is then proving that μ^+, as stated above, is indeed an 2_Σ-valuation. Formally, this assertion means that for each $\# \in \Sigma^{(n)}$, with $n \in \omega$,

$$\mu^+(\#(A_1, \ldots, A_n)) = 1 \text{ iff } \#^{2_\Sigma}(\mu^+(A_1), \ldots, \mu^+(A_n)) = 1, \tag{2.1}$$

where $\#^{2_\Sigma}$ is the interpretation of the connective $\#$ in 2_Σ. The right-hand side of that condition is equivalent to the meta-disjunction of the k meta-conjunctions of the form $\mu^+(A_1) = \alpha_1^j$ and $\ldots$ and $\mu^+(A_n) = \alpha_n^j$ corresponding to each of the k determinants $\langle \alpha_1^j, \ldots, \alpha_n^j, 1\rangle$ of the truth-function of $\#^{2_\Sigma}$, where $1 \leq j \leq k$. Because $\mu^+(A) = 1$ iff $A \in \Gamma^+$, proving assertion 2.1 is equivalent to proving the property

$$\#(A_1, \ldots, A_n) \in \Gamma^+ \tag{\#}$$

$$\text{iff}$$

$$(A_1 \in_1^1 \Gamma^+ \text{ and } \ldots \text{ and } A_n \in_n^1 \Gamma^+) \text{ or } \ldots \text{ or } (A_1 \in_1^k \Gamma^+ \text{ and } \ldots \text{ and } A_n \in_n^k \Gamma^+),$$

where $\in_i^j = \in$ if $\alpha_i^j = 1$, and $\in_i^j = \notin$ otherwise. Notice that if $\#$ is nullary and $\#^{2_\Sigma} = 1$, the property (#) reduces to $\# \in \Gamma^+$, and, in the case $\#^{2_\Sigma} = 0$, it becomes $\# \notin \Gamma^+$. Depending on the truth-table of the connective $\#$ in 2_Σ, the expression of property (#) can be highly simplified, easing the path to prove it.

Example 2.7.1. Let us find the completeness property for $\vee$, namely $(\vee)$. By inspecting its truth-table (see Example 2.5.1), the determinants of interest are $\langle 0, 1, 1\rangle$, $\langle 1, 0, 1\rangle$ and

$\langle 1,1,1\rangle$. The specialization of the expression in property (#) for the case of $\vee$ gives us

$$\mathrm{A}_1 \vee \mathrm{A}_2 \in \Gamma^+$$
$$\text{iff}$$
$$(\mathrm{A}_1 \notin \Gamma^+ \text{ and } \mathrm{A}_2 \in \Gamma^+) \text{ or } (\mathrm{A}_1 \in \Gamma^+ \text{ and } \mathrm{A}_2 \notin \Gamma^+) \text{ or } (\mathrm{A}_1 \in \Gamma^+ \text{ and } \mathrm{A}_2 \in \Gamma^+)$$

that can be simplified to

$$\mathrm{A}_1 \vee \mathrm{A}_2 \in \Gamma^+ \text{ iff } \mathrm{A}_1 \in \Gamma^+ \text{ or } \mathrm{A}_2 \in \Gamma^+.$$

Since proving the aforementioned property for every connective in the language of the calculus under consideration means that μ^+ is a $\mathbb{2}_\Sigma$-valuation, and, as it was already shown, $\mu^+(\Gamma) \subseteq \{1\}$ but $\mu^+(\mathrm{Z}) = 0$, the conclusion is that $\Gamma \not\vdash_{\mathbb{2}_\Sigma} \mathrm{Z}$, finishing the completeness proof. This method is repeatedly applied in Chapter 4 to prove the completeness of the proposed calculus for each of the fragments of Classical Logic under study.

Remark 2.7.2. The technique just described can be naturally extended for logical matrices with more than two values. In such case, we have to look for an appropriate mapping to play the role of μ^+ and to express property (#) in terms of the set of designated values of the matrix.

2.8 Axiomatic Expansion Lemma

Numerous fragments of Classical Logic, under the perspective of 2-matrices, result from other fragments by adding the nullary operation $\top$ to their base set of operations. If the properties necessary for completeness of a calculus in the reducts are preserved in the expansions, then axiomatizing the calculus of the expansion becomes an easy task. The main goal of this section is to show that sequent-style rules are preserved in countable axiomatic expansions, a result from which the completeness of all calculus resulting from expansions by the constant $\top$ trivially follows (see [5] for a more general version).

Before getting into it, some preliminary results are due. The first of them shows that the notion of reasoning with new axioms in an expanded language can be equivalently expressed by using them either as assumptions in the least expanded logic or incorporating them by means of the axiomatic expansion notion.

Lemma 2.8.1. *Let Σ and Σ^+ be signatures such that Σ^+ expands Σ, and let $\vdash$ be a consequence relation over Σ. Then, where $\Lambda \subseteq L^P_{\Sigma^+}$ and $\vdash_+$ is the smallest expansion of*

$\vdash$ *to* $L^P_{\Sigma^+}$*, the following equivalence holds:*

$$\text{for all } \Gamma \cup \{C\} \subseteq L^P_{\Sigma^+}.\ \Gamma \vdash_{+\Lambda} C \text{ iff } \Gamma, \mathcal{S}(\Lambda) \vdash_+ C, \qquad (*^a)$$

where $\mathcal{S}(\Lambda) = \{A^\sigma \mid A \in \Lambda, \sigma \in \mathsf{Sb}(\mathbf{L}^P_{\Sigma^+})\}$.

Proof. Consider the relation $\vdash_0$ such that $\Gamma \vdash_0 B$ if, and only if, $\Gamma, \mathcal{S}(\Lambda) \vdash_+ B$, where $\vdash_+$ is the least expansion of $\vdash$ in Σ^+. Our goal now is to show that $\vdash_0$ is a consequence relation over Σ^+ that expands $\vdash$. Notice that if $\Gamma \vdash B$, then $\Gamma \vdash_+ B$, which, by (M), yields $\Gamma, \mathcal{S}(\Lambda) \vdash_+ B$, thus $\Gamma \vdash_0 B$, so $\vdash_0$ expands $\vdash$. Also, by (R) and (M), we have $\{B\}, \mathcal{S}(\Lambda) \vdash_+ B$, then $\{B\} \vdash_0 B$, thus $\vdash_0$ respects the property (R). Moreover, if $\Theta \subseteq \Gamma$ and $\Theta \vdash_0 B$, then, by (M), $\Theta, \Gamma, \mathcal{S}(\Lambda) \vdash_+ B$, meaning that $\Gamma, \mathcal{S}(\Lambda) \vdash_+ B$, thus $\Gamma \vdash_+ B$, so that $\vdash_0$ respects property (M). Properties (T) and (S) also follow from those in $\vdash_+$. Therefore, $\vdash_0$ is a consequence relation over Σ^+ that expands $\vdash$. Then, proving $(*^a)$ is equivalent to proving that $\vdash_0 = \vdash_{+\Lambda}$. With this aim, let $\vdash'$ be an expansion of $\vdash$ such that $\vdash' A$ for all $A \in \Lambda$. By definition of $\vdash_0$, if $\Gamma \vdash_0 B$, then $\Gamma, \mathcal{S}(\Lambda) \vdash_+ B$. Hence, because $\vdash_+$ is the least expansion of $\vdash$ on $L^P_{\Sigma^+}$, we have (a): $\Gamma, \mathcal{S}(\Lambda) \vdash' B$. Since (b): $\vdash' A$ for all $A \in \Lambda$, by (T) from (a) and (b), we get $\Gamma \vdash' B$, so $\vdash_0 \subseteq \vdash'$, proving that $\vdash_0$ is the least expansion of $\vdash$ that incorporates the formulas in Λ as theorems, that is, $\vdash_0 = \vdash_{+\Lambda}$. □

The next two lemmas establish relations between substitutions in the original and in the expanded language, and in the original and in the least expanded logic, respectively. Henceforth, we use sk_Σ to denote the skeleton function given in Example 2.2.3, with the underlying enumeration of the formulas in $L^P_{\Sigma \setminus \Sigma^+}$ being $\{Q_i\}_{i \in \omega}$.

Lemma 2.8.2. *For signatures* Σ *and* Σ^+*, such that* $\Sigma \subseteq \Sigma^+$*, and a substitution* $e^+ \in \mathsf{Sb}(\mathbf{L}^P_{\Sigma^+})$*, there is a substitution* $e \in \mathsf{Sb}(\mathbf{L}^P_\Sigma)$ *such that:*

$$\mathsf{sk}_\Sigma \circ e^+ = e \circ \mathsf{sk}_\Sigma.$$

Proof. Consider arbitrary signatures Σ and Σ^+ such that $\Sigma \subseteq \Sigma^+$, and a substitution $e^+ \in \mathsf{Sb}(\mathbf{L}^P_{\Sigma^+})$. Then, let $e = \mathsf{sk}_\Sigma \circ e^+ \circ \mathsf{sk}_\Sigma^{-1}$. The equation of this lemma obviously holds and, because sk_Σ and its inverse satisfy the homomorphism condition (see the definition of skeleton in Section 2.2), e is a substitution. □

Lemma 2.8.3. *Let* Σ *and* Σ^+ *be signatures such that* $\Sigma \subseteq \Sigma^+$*, and let* $\vdash$ *be a consequence relation over* Σ*, with* $\vdash_+$ *being its least expansion over* Σ^+*. Then, for* $\Gamma \cup \{A\} \subseteq L^P_{\Sigma^+}$*, the following holds:*

$$\Gamma \vdash_+ A \text{ iff } \mathsf{sk}_\Sigma(\Gamma) \vdash \mathsf{sk}_\Sigma(A). \qquad (*^b)$$

Proof. Let $\vdash_0$ be a relation such that $\Gamma \vdash_0 \mathrm{A}$ if, and only if, $\mathsf{sk}_\Sigma(\Gamma) \vdash \mathsf{sk}_\Sigma(\mathrm{A})$. We proceed by proving that $\vdash_0 = \vdash_+$. First, notice that $\vdash_0$ is a consequence relation because:

- (R) follows from the fact that $\mathsf{sk}_\Sigma(\mathrm{A}) \vdash \mathsf{sk}_\Sigma(\mathrm{A})$, since $\vdash$ is a consequence relation;
- (M) follows since, if $\Theta \vdash_0 \mathrm{A}$, then $\mathsf{sk}_\Sigma(\Theta) \vdash \mathsf{sk}_\Sigma(\mathrm{A})$, and $\mathsf{sk}_\Sigma(\Theta), \mathsf{sk}_\Sigma(\Gamma) \vdash \mathsf{sk}_\Sigma(\mathrm{A})$ (because $\vdash$ respects (M)), but, if $\Theta \subseteq \Gamma$, then $\mathsf{sk}_\Sigma(\Theta) \subseteq \mathsf{sk}_\Sigma(\Gamma)$, so $\mathsf{sk}_\Sigma(\Gamma) \vdash \mathsf{sk}_\Sigma(\mathrm{A})$, and $\Gamma \vdash_0 \mathrm{A}$;
- (T) follows because, if $\Gamma, \Theta \vdash_0 \mathrm{A}$ and $\Gamma \vdash_0 \mathrm{B}$ for each $\mathrm{B} \in \Theta$, then $\mathsf{sk}_\Sigma(\Gamma), \mathsf{sk}_\Sigma(\Theta) \vdash \mathsf{sk}_\Sigma(\mathrm{A})$ and $\mathsf{sk}_\Sigma(\Gamma) \vdash \mathsf{sk}_\Sigma(\mathrm{B})$ for each $\mathrm{B} \in \Theta$, so $\mathsf{sk}_\Sigma(\Gamma) \vdash \mathrm{B}'$, for each $\mathrm{B}' \in \mathsf{sk}_\Sigma(\Theta)$, leading to $\mathsf{sk}_\Sigma(\Gamma) \vdash \mathsf{sk}_\Sigma(\mathrm{A})$ (by (T) of $\vdash$), thus $\Gamma \vdash_0 \mathrm{A}$; and
- (S) follows from Lemma 2.8.2.

Notice that sk^*_Σ, the restriction of sk_Σ to L_Σ, is a substitution (this follows directly from the definition of a skeleton given in Section 2.2), so, if $\Gamma \vdash \mathrm{A}$, then, by (S), $\mathsf{sk}^*_\Sigma(\Gamma) \vdash \mathsf{sk}^*_\Sigma(\mathrm{A})$, and, by the definition of $\vdash_0$, $\Gamma \vdash_0 \mathrm{A}$, thus $\vdash_0$ expands $\vdash$.

Now, let $\vdash'$ be a consequence relation over Σ^+ that expands $\vdash$. For $\Gamma \cup \{\mathrm{A}\} \subseteq L^P_{\Sigma+}$, suppose that $\Gamma \vdash_0 \mathrm{A}$, so by the definition of $\vdash_0$, $\mathsf{sk}_\Sigma(\Gamma) \vdash \mathsf{sk}_\Sigma(\mathrm{A})$. Define the function $b : L^P_{\Sigma+} \to L^P_{\Sigma+}$ such that:

$$b(\mathrm{C}) = \begin{cases} \mathrm{p}_i & \text{if } \mathrm{C} = \mathrm{p}_{2i}, \\ \mathrm{Q}_i & \text{if } \mathrm{C} = \mathrm{p}_{2i+1}, \\ \#(b(\mathrm{C}_1), \ldots, b(\mathrm{C}_n)) & \text{if } \mathrm{C} = \#(\mathrm{C}_1, \ldots, \mathrm{C}_n). \end{cases}$$

Notice that the third case in this definition makes b a substitution on $\mathbf{L}^P_{\Sigma+}$. Because $\vdash \subseteq \vdash'$, we have $\mathsf{sk}_\Sigma(\Gamma) \vdash' \mathsf{sk}_\Sigma(\mathrm{A})$, and, since $b \in \mathsf{Sb}(\mathbf{L}^P_{\Sigma+})$ and $\vdash'$ is a consequence relation, $b(\mathsf{sk}_\Sigma(\Gamma)) \vdash' b(\mathsf{sk}_\Sigma(\mathrm{A}))$, thus $\Gamma \vdash' \mathrm{A}$, given that $b \circ \mathsf{sk}_\Sigma$ is the identity on $L^P_{\Sigma+}$ (by restricting b to L^P_Σ). This shows that $\vdash_0 \subseteq \vdash'$, thus $\vdash_0 = \vdash_+$. ☐

It is worth noting that (for sk^*_Σ as defined in the lemma above), because sk^*_Σ is a substitution on $\mathbf{L}^P_\Sigma$:

$$\text{if } d \in \mathsf{Sb}(\mathbf{L}^P_\Sigma) \text{ then } d \circ \mathsf{sk}^*_\Sigma \in \mathsf{Sb}(\mathbf{L}^P_\Sigma). \qquad (*^d)$$

Theorem 2.8.4. *A sequent-style rule that holds in a consequence relation $\vdash$ over Σ holds in each (countable) axiomatic expansion of $\vdash$.*

Proof. Let Σ be an arbitrary signature, and consider a sequent-style rule over Σ having the form $\Delta_1 \succ C_1, \ldots, \Delta_n \succ C_n / \Delta \succ C$, and assume that it holds in $\vdash$, that is, for all $\sigma \in \mathsf{Sb}(\mathbf{L}^P_\Sigma)$ and $\Gamma \subseteq L^P_\Sigma$:

$$\text{if } \Gamma, \Delta_1^\sigma \vdash C_1^\sigma \text{ and } \ldots \text{ and } \Gamma, \Delta_n^\sigma \vdash C_n^\sigma \text{ then } \Gamma, \Delta^\sigma \vdash C^\sigma. \qquad (\star)$$

Now, let Σ^+ be a signature that expands Σ and let $\Lambda \subseteq L_{\Sigma^+}$. In order to complete this proof, the following needs to be proved, for all $\sigma \in \mathsf{Sb}(\mathbf{L}^P_{\Sigma^+})$ and $\Gamma \subseteq L^P_{\Sigma^+}$:

$$\text{if } \Gamma, \Delta_1^\sigma \vdash_{+\Lambda} C_1^\sigma \text{ and } \ldots \text{ and } \Gamma, \Delta_n^\sigma \vdash_{+\Lambda} C_n^\sigma \text{ then } \Gamma, \Delta^\sigma \vdash_{+\Lambda} C^\sigma \qquad (\star^+)$$

Accordingly, let $\sigma \in \mathsf{Sb}(\Sigma^+)$ and $\Gamma \subseteq L_{\Sigma^+}$. Suppose that $\Gamma, \Delta_i^\sigma \vdash_{+\Lambda} C_i^\sigma$, for all $1 \leq i \leq n$. Then the following reasoning proves the desired result (we change C^σ and Δ^σ to $\sigma(C)$ and $\sigma(\Delta)$, respectively, for better legibility):

(1)	$\Gamma, \sigma(\Delta_i) \vdash_{+\Lambda} \sigma(C_i)$	Assumptions, all $1 \leq i \leq n$
(2)	$\mathcal{S}(\Lambda), \Gamma, \sigma(\Delta_i) \vdash_{+} \sigma(C_i)$	$1 *^a$
(3)	$\mathsf{sk}_\Sigma(\mathcal{S}(\Lambda)), \mathsf{sk}_\Sigma(\Gamma), \mathsf{sk}_\Sigma(\sigma(\Delta_i)) \vdash \mathsf{sk}_\Sigma(\sigma(C_i))$	$2 *^b$
(4)	$\mathsf{sk}_\Sigma(\mathcal{S}(\Lambda)), \mathsf{sk}_\Sigma(\Gamma), \bar{\sigma}(\mathsf{sk}_\Sigma(\Delta_i)) \vdash \bar{\sigma}(\mathsf{sk}_\Sigma(C_i))$	3 Lemma 2.8.2 ($e^+ = \sigma$ and $e = \bar{\sigma}$)
(5)	$\mathsf{sk}_\Sigma(\mathcal{S}(\Lambda)), \mathsf{sk}_\Sigma(\Gamma), \bar{\sigma}(\mathsf{sk}_\Sigma(\Delta)) \vdash \bar{\sigma}(\mathsf{sk}_\Sigma(C))$	$4 *^d$ and $\star$
(6)	$\mathsf{sk}_\Sigma(\mathcal{S}(\Lambda)), \mathsf{sk}_\Sigma(\Gamma), \mathsf{sk}_\Sigma(\sigma(\Delta)) \vdash \mathsf{sk}_\Sigma(\sigma(C))$	5 Lemma 2.8.2 ($e^+ = \sigma$ and $e = \bar{\sigma}$)
(7)	$\mathcal{S}(\Lambda), \Gamma, \sigma(\Delta) \vdash_{+} \sigma(C)$	$6 *^b$
(8)	$\Gamma, \sigma(\Delta) \vdash_{+\Lambda} \sigma(C)$	$7 *^a$

□

Corollary 2.8.4.1. *If $\mathcal{B}_\Sigma$ is axiomatized by $\mathscr{B}_\Sigma$ as a consequence of sequent-style rules, then $\mathcal{B}_{\Sigma,\top}$ is axiomatized by the calculus resulting from the rules of $\mathscr{B}_\Sigma$ together with the nullary rule* $(\mathsf{t}_1)\ /\top$.

Proof. Given Theorem 2.8.4, we only have to show that the consequence relation $\vdash_{\mathscr{B}_{\Sigma,\top}}$ is the axiomatic expansion of $\vdash_{\mathscr{B}_\Sigma}$ determined by $\{\top\}$, i.e. the least expansion of $\vdash_{\mathscr{B}_\Sigma}$ having $\top$ as theorem. First of all, notice that $\vdash_{\mathscr{B}_{\Sigma,\top}} \top$, given the presence of rule t_1. Also, it is clear that $\vdash_{\mathscr{B}_{\Sigma,\top}}$ expands $\vdash_{\mathscr{B}_\Sigma}$, because $\mathscr{B}_{\Sigma,\top}$ has all the rules of $\mathscr{B}_\Sigma$. Now it remains to show that $\vdash_{\mathscr{B}_{\Sigma,\top}}$ is the least expansion of $\vdash_{\mathscr{B}_\Sigma}$ having $\top$ as theorem. For that, suppose that $\vdash_{\mathscr{B}_\Sigma} \subseteq \vdash'$, for some $\vdash'$ having $\top$ as theorem. We will show that $\vdash_{\mathscr{B}_{\Sigma,\top}} \subseteq \vdash'$ by induction on a derivation in $\mathscr{B}_{\Sigma,\top}$. So suppose that $\Gamma \vdash_{\mathscr{B}_{\Sigma,\top}} A$ and that this is witnessed by a derivation $A_1, \ldots, A_n = A$. Let $P(i)$ mean the consecution $\Gamma \vdash' A_i$, so that $P(n)$

is what we want for the present proof. For the base case, we have two possibilities: (i) $\mathrm{A}_1 \in \Gamma$ or (ii) A_1 is an axiom instance of $\mathscr{B}_{\Sigma,\top}$. For (i), apply (R) and (M) to get $\Gamma \vdash' \mathrm{A}_1$. For (ii), it is clear that if $\vdash_{\mathscr{B}_\Sigma} \mathrm{A}_1$ then $\vdash' \mathrm{A}_1$, since $\vdash'$ expands $\vdash_{\mathscr{B}_\Sigma}$, and, in case $\mathrm{A}_1 = \top$, we know that $\vdash' \top$, so this possibility also lead to $\Gamma \vdash' \mathrm{A}_1$ by (M). For the inductive step, suppose that $P(j)$ holds for all $1 \leq j < k$ with $k > 1$. Then we have three possibilities for A_k: (i) and (ii), whose proofs are the same as before, and (iii) A_k results from the instance $\langle \mathrm{A}_{k_1}, \ldots, \mathrm{A}_{k_m}, \mathrm{A}_k \rangle$, for $k_l < k$ and $1 \leq l \leq m$ of an m-ary rule of $\mathscr{B}_{\Sigma,\top}$. Since this rule must also be a rule of $\mathscr{B}_\Sigma$, we have $\mathrm{A}_{k_1}, \ldots, \mathrm{A}_{k_m} \vdash' \mathrm{A}$, thus, by (M), (a): $\Gamma, \mathrm{A}_{k_1}, \ldots, \mathrm{A}_{k_m} \vdash' \mathrm{A}_k$. Then, from (a) and the induction hypothesis, we get $\Gamma \vdash' \mathrm{A}_k$ by (T). $\square$

3 Hilbert calculi on Lean

Lean is a programming language and a theorem prover that aims to support both interactive and automated theorem proving in a general and unified framework [1]. It is based on a version of dependent type theory called Calculus of Constructions [6], which can express complex mathematical assertions, specify hardware and software, and reason naturally and uniformly about them. This work explores the capacity of Lean to accommodate the specification of axiomatic systems and verify that every claim about them is justified by an appeal to prior definitions and theorems. Henceforth, we will use Example 2.3.1, which presents a well-known axiomatization for Classical Logic, to illustrate the task of specifying a Hilbert calculus and proving some properties about it. In Section 4, we will apply the same strategies to specify the proposed calculi for the main fragments of Classical Logic.

As we know from Section 2, the definition of a Hilbert calculus demands a language, which is specified by means of a signature made of symbols — the connectives of the language — with an associated arity, and a set of rules of inference. Therefore, before proving anything about a calculus, we need to give its specification in Lean in terms of its connectives and inference rules.

In Lean, propositions — the elements of a language — are treated as objects of the built-in type `Prop`. So, for example, the expression `a → (b ∨ c)` in Lean denotes an object of type `Prop`. We can use the command `#check` to verify the type of an object, so `#check(a → (b ∨ c))`, given the appropriate declarations of the variables, outputs `a → b ∨ c : Prop`. Since connectives in a language are operations that transform formulas into a new formula, in Lean they are implemented as functions over the type `Prop`. We will use the keyword `constant` to introduce new function symbols in the working environment and, in order to declare a function that takes an argument of type `A` and transforms it into an object of type `B`, we use the construction `A → B`. So, the way we declare the connectives for our example is given below.

```
-- conjunction
```

```
constant and : Prop → Prop → Prop
-- disjunction
constant or : Prop → Prop → Prop
-- implication
constant imp : Prop → Prop → Prop
-- negation
constant neg : Prop → Prop
-- top
constant top : Prop
-- bottom
constant bot : Prop
```

In addition, we can let the expressions with binary connectives be given in infix notation by using the `notation` construct:

```
notation a ‘or‘ b := or a b
notation a ‘imp‘ b := imp a b
notation a ‘and‘ b := and a b
```

Now that we have the language for our calculus, we need to specify its rules. Rules are also seen as functions in Lean, but this time defined in terms of dependent types, so that we can apply them to any formulas that obey the rule format (remember that a rule is presented schematically, but is actually an infinite set of tuples determined by substitutions over its schema). Such types are called Pi types, whose detailed exposition is not of interest to us here, and they have a very convenient notation in Lean, using the symbol ∀. Because we want to give names to the rules so that we can use them in proofs, we have to define new symbols in the environment, what is possible via the `constant` construct, as we did for defining the connectives. Lean, however, offers the keyword `axiom` with the same purpose, and thus we can make our specification closer to the typical mathematical jargon.

Below we have the specification of the rules (and axioms) for our example. Notice that the implementation of cl_1 (the rule of *modus ponens*) is a function that accepts an object of type `a` and another object of type `a imp b` and gives an object of type `b`. A common interpretation for such objects is to consider them as proofs of the formulas corresponding to their types.

```
axiom cl₁ : ∀ {a b : Prop}, a → a imp b → b
axiom cl₂ : ∀ {a b : Prop}, a imp (b imp a)
axiom cl₃ : ∀ {a b c : Prop}, (a imp (b imp c)) imp ((a imp b) imp (a imp c))
axiom cl₄ : ∀ {a b : Prop}, (a and b) imp a
```

```
axiom cl5 : ∀ {a b : Prop}, (a and b) imp b
axiom cl6 : ∀ {a b : Prop}, a imp (b imp (a and b))
axiom cl7 : ∀ {a b : Prop}, a imp (a or b)
axiom cl8 : ∀ {a b : Prop}, b imp (a or b)
axiom cl9 : ∀ {a b c : Prop}, (a imp c) imp ((b imp c) imp ((a or b) imp c))
axiom cl10 : ∀ {a b : Prop}, (a imp b) imp ((a imp (neg b)) imp (neg a))
axiom cl11 : ∀ {a : Prop}, (neg (neg a)) imp a
axiom cl12 : top
axiom cl13 : ∀ {a : Prop}, bot imp a
```

At this point, we have our system fully specified in Lean. We are now ready to prove properties about it. For example, we want to show that $A \to A$, for each formula A, is a theorem in the system under discussion. For that, we have to present a sequence of formulas (a derivation) ending with $A \to A$, where each of its elements is either an instance of an axiom or results from an application of a non-nullary rule of the calculus to earlier formulas in the sequence. At each step in this derivation, we commonly want to justify what rule (or axiom) and what formulas (if any) were used.

In Lean, when we want to prove a property like this without the need to name it, we can use the keyword `example` to establish the property, and each step in the derivation uses the keyword `have` followed by the formula we want to derive, and the keyword `from` to indicate the rule (or axiom) and the formulas (if any) we used to derive it. This last construct is actually the application of a rule to some formulas, in the functional sense. That said, see below the derivation of `a imp a` in Lean:

```
example {a : Prop} : a imp a :=
  have h1 : (a imp ((a imp a) imp a)) imp ((a imp (a imp a)) imp (a imp a)), from cl3,
  have h2 : a imp ((a imp a) imp a), from cl2,
  have h3 : ((a imp (a imp a)) imp (a imp a)), from cl1 h2 h1,
  have h4 : a imp (a imp a), from cl2,
  show a imp a, from cl1 h4 h3
```

Notice that we can easily translate this code to the typical form of a derivation tree:

$$\dfrac{\dfrac{}{A \to (A \to A)}\,\mathsf{cl}_2 \qquad \dfrac{\dfrac{}{(A \to ((A \to A) \to A)) \to ((A \to (A \to A)) \to (A \to A))}\,\mathsf{cl}_3 \qquad \dfrac{}{A \to ((A \to A) \to A)}\,\mathsf{cl}_2}{(A \to (A \to A)) \to (A \to A)}\,\mathsf{cl}_1}{A \to A}\,\mathsf{cl}_1$$

Moreover, in this work, we will be often required to prove the derivability of non-nullary rules in a calculus, and, in addition, to use these rules in other derivations, requiring thus name for them. The way we do this in Lean is very similar to the above example, but, instead of using the `example` construct, we use `theorem`. Since we want a non-nullary rule, we will also declare parameters for the property, so that we can use them in the derivation. Just to give an example, suppose that our task is to show the derivability of the following rule in our calculus for Classical Logic:

$$\frac{A \to B \quad B \to C}{A \to C}\ \mathsf{cl}_{14}$$

Then we can see the Lean code for it as a function taking as arguments a proof of `a imp b` and a proof of `b imp c`, and producing a proof of `a imp c` by means of a derivation in the sense of the previous example. Section 4 is full of derivations like this, since we need to show the derivability of rules that are important for proving the completeness of most of the calculi we are going to deal with. Below we present the proof of the derivability of cl_{14} in Lean code, which can also be easily translated into a derivation tree, sketched in the sequel.

```
theorem cl14 {a b c : Prop} (h1 : a imp b) (h2 : b imp c) : a imp c :=
  have h3 : (a imp (b imp c)) imp ((a imp b) imp (a imp c)), from cl3,
  have h4 : (b imp c) imp (a imp (b imp c)), from cl2,
  have h5 : a imp (b imp c), from cl1 h2 h4,
  have h6 : (a imp b) imp (a imp c), from cl1 h5 h3,
  show a imp c, from cl1 h1 h6
```

$$\dfrac{\dfrac{\dfrac{\dfrac{}{(B \to C) \to (A \to (B \to C))}\ \mathsf{cl}_2 \qquad \dfrac{}{B \to C}}{A \to (B \to C)}\ \mathsf{cl}_1 \qquad \dfrac{}{(A \to (B \to C)) \to ((A \to B) \to (A \to C))}\ \mathsf{cl}_3}{(A \to B) \to (A \to C)}\ \mathsf{cl}_1 \qquad \dfrac{}{A \to B}}{A \to C}\ \mathsf{cl}_1$$

We finish this exposition with the representation in Lean of sequent-style rules. Such kind of rules can be useful when we need meta-properties in a proof of derivability, that is, when we reason in terms of properties regarding the associated consequence relation. Section 4.7 is full of examples of derivability proofs using the rules given in Example 2.2.4 together with weaker versions of property $\delta_\vee$ (see Lemma 4.7.3). The implementation in Lean of such rules uses again functions on dependent types, but now taking as arguments functions representing each sequent in the following way: if $A_1, \ldots, A_n \succ B$ is a sequent, then we let the functional type `a1` → ... → `an` → `b` be the type of the objects representing that sequent. To illustrate this, the Lean code for the aforementioned rules is:

```
-- rules holding in every consequence relation
axiom R : ∀ {a : Prop}, a → a
axiom M₁ : ∀ {a b c : Prop}, (a → b) → (c → a → b)
axiom T₁ : ∀ {a b c : Prop}, (a → b) → (b → c) → (a → c)
axiom T₂ : ∀ {a b c d : Prop}, (d → a → b) → (d → b → c) → (d → a → c)
-- weaker versions of δ_or
axiom δ_or₁ : ∀ {a b c : Prop}, (a → c) → (b → c) → ((a or b) → c)
axiom δ_or₂ : ∀ {a b c d : Prop}, (d → a → c) → (d → b → c) → (d → (a or b) → c)
```

4 The fragments and the calculi

This chapter is devoted to the detailed presentation of a Hilbert calculus for each of the main fragments of Classical Logic. Figure 2 shows a version of Post's lattice highlighting only the fragments of interest for this work — those in the finite portion of the structure — and exhibiting some groups (determined by the black lines) in which such fragments are organized. We intend here to cover the lattice in the ascending order of the numbers corresponding to each group. Each group is covered bottom-up, with its members being developed in one or more sections of this chapter. A fragment together with its expansions by constants are often grouped in the same section, since, in general, their calculi differ by minor changes.

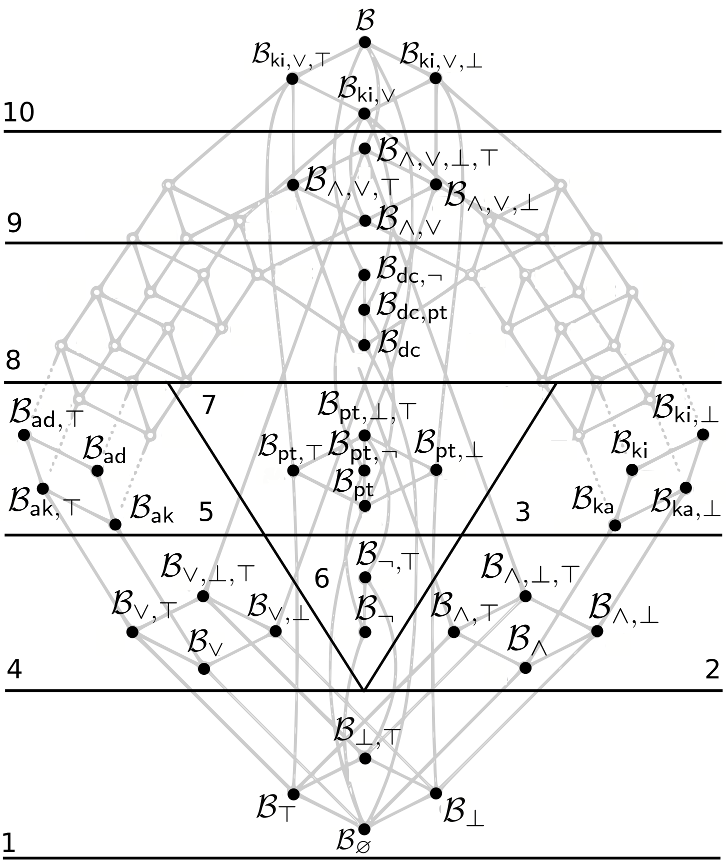

Figure 2: Main fragments of Classical Logic in Post's lattice.

4.1 $\boldsymbol{\mathcal{B}}_\varnothing$, $\boldsymbol{\mathcal{B}}_\top$, $\boldsymbol{\mathcal{B}}_\bot$, $\boldsymbol{\mathcal{B}}_{\bot,\top}$

The peculiar fragment of Classical Logic induced by the logical matrix $\mathbb{2}_\varnothing$ is axiomatized by the equally peculiar Hilbert calculus $\mathscr{B}_\varnothing$ whose set of rules is empty. In order to understand why no rules are necessary in this case, first notice that, in the absence of connectives, every formula of $\mathbf{L}_\varnothing$ is a propositional variable, that is, where P is the set of propositional variables, $L_\varnothing = P$. Hence, the set of $\mathbb{2}_\varnothing$-valuations is the set of all functions from P into $\{0,1\}$. Because of that, whenever $\Gamma \cup \{\mathrm{p}\} \subseteq L_\varnothing$ and $\Gamma \vdash_{\mathbb{2}_\varnothing} \mathrm{p}$, the variable p must be in Γ, for otherwise we could take any valuation that assigns the value 1 to every variable in Γ and 0 to p. Since $\mathrm{p} \in \Gamma$, properties (R) and (M) imply that $\Gamma \vdash_{\mathscr{B}_\varnothing} \mathrm{p}$, the desired completeness result. Notice that any calculus over $\mathbf{L}_\varnothing$ would be adequate in this case, but the least one that is sound (clearly by vacuity) with respect to $\mathbb{2}_\varnothing$ is $\mathscr{B}_\varnothing$.

We proceed now to the fragments $\mathcal{B}_\top$, $\mathcal{B}_\bot$ and $\mathcal{B}_{\bot,\top}$, whose axiomatizations are as simple as their languages. The calculus $\mathscr{B}_\top$, presented below, will be used throughout the present chapter (and at the end of the present section!) in mergings that axiomatize expansions of other fragments of Classical Logic by the constant $\top$, in view of Corollary 2.8.4.1.

Hilbert Calculus 2. $\mathscr{B}_\top$

$$\frac{}{\top}\ \mathsf{t}_1$$

Theorem 4.1.1. *The calculus $\mathscr{B}_\top$ is sound with respect to the matrix $\mathbb{2}_\top$.*

Proof. There is only one rule and its conclusion is evaluated to 1 for any $\mathbb{2}_\top$-valuation, what guarantees the soundness of the calculus under discussion. □

Theorem 4.1.2. *The calculus $\mathscr{B}_\top$ is complete with respect to the matrix $\mathbb{2}_\top$.*

Proof. Consider the procedure described in Section 2.7, let $\Gamma \cup \{\mathrm{Z}\} \subseteq L_\top$, such that $\Gamma \nvdash_{\mathscr{B}_\top} \mathrm{Z}$ and take a Z-maximal theory $\Gamma^+ \supseteq \Gamma$. Since $\top$ is the sole connective in this language, and $v(\top) = 1$ for any $\mathbb{2}_\top$-valuation, proving completeness in this case is a matter of showing that

$$\top \in \Gamma^+, \tag{$\top$}$$

by an specialization of property (#). Notice that $\Gamma^+ \vdash_{\mathscr{B}_\top} \top$, by t_1 and (M). Then, by using Corollary 2.6.1.1, we have $\top \in \Gamma^+$. □

Hilbert Calculus 3. $\mathscr{B}_\perp$

$$\frac{\perp}{A}\ \mathsf{b}_1$$

Theorem 4.1.3. *The calculus $\mathscr{B}_\perp$ is sound with respect to the matrix $\mathbb{2}_\perp$.*

Proof. There is only one rule and its sole premiss is evaluated to 0 for any $\mathbb{2}_\perp$-valuation; therefore the calculus is sound with respect to $\mathbb{2}_\perp$. □

Theorem 4.1.4. *The calculus $\mathscr{B}_\perp$ is complete with respect to the matrix $\mathbb{2}_\perp$.*

Proof. Considering the procedure described in Section 2.7, let $\Gamma \cup \{Z\} \subseteq L_\perp$, such that $\Gamma \nvdash_{\mathscr{B}_\perp} Z$ and take a Z-maximal theory $\Gamma^+ \supseteq \Gamma$. Since $\perp$ is the sole connective in this language, proving completeness amounts to showing the following specialization of Property 2.1:

$$\perp \notin \Gamma^+. \tag{$\perp$}$$

The proof goes by contradiction: assume that $\perp \in \Gamma^+$. Then, by Corollary 2.6.1.1, $\Gamma^+ \vdash_{\mathscr{B}_\perp} \perp$. The instance $\langle \perp, Z\rangle$ of rule b_1, alongside with (M), implies that $\Gamma^+, \perp \vdash_{\mathscr{B}_\perp} Z$. Property (T) applied to the mentioned assertions gives $\Gamma^+ \vdash_{\mathscr{B}_\perp} Z$, contradicting the fact that $\Gamma^+ \nvdash_{\mathscr{B}_\perp} Z$. □

Remark 4.1.1. Notice that the presence of the rule b_1 in a calculus having $\perp$ in its language is enough to guarantee the completeness property ($\perp$).

The expansion $\mathcal{B}_{\top,\perp}$ is axiomatized by the calculus below, given the preservation results for $\top$-expansions presented in Corollary 2.8.4.1:

Hilbert Calculus 4. $\mathscr{B}_{\perp,\top}$

$$\mathscr{B}_\perp \quad \mathscr{B}_\top$$

4.2 $\mathcal{B}_\wedge, \mathcal{B}_{\wedge,\top}, \mathcal{B}_{\wedge,\perp}, \mathcal{B}_{\wedge,\perp,\top}$

The calculus for $\mathcal{B}_\wedge$ proposed below reflects the behaviour of $\wedge^2$ and, as we will see, its rules are all we need to prove the completeness property ($\wedge$), obtained by specializing property (#) for the case of $\wedge$.

Hilbert Calculus 5. $\mathscr{B}_\wedge$

$$\frac{A \quad B}{A \wedge B}\,\mathsf{c}_1 \qquad \frac{A \wedge B}{A}\,\mathsf{c}_2 \qquad \frac{A \wedge B}{B}\,\mathsf{c}_3$$

Theorem 4.2.1. *The calculus $\mathscr{B}_\wedge$ is sound with respect to the matrix $\mathfrak{2}_\wedge$.*

Proof. Let $A, B \in L_\wedge$. Suppose that $\langle A, B, A \wedge B\rangle \in \mathsf{c}_1$. Let v be a $\mathfrak{2}_\wedge$-evaluation, such that $v(A) = 1$ and $v(B) = 1$. By the truth-table of $\wedge$ in $\mathfrak{2}_\wedge$, $v(A \wedge B) = 1$. For rule c_2, suppose that $\langle A \wedge B, A\rangle \in \mathsf{c}_2$ and that $v(A \wedge B) = 1$. Then the truth-table of $\wedge$ in $\mathfrak{2}_\wedge$ imposes that $v(A) = 1$. The proof for rule c_3 is analogous. □

Theorem 4.2.2. *The calculus $\mathscr{B}_\wedge$ is complete with respect to the matrix $\mathfrak{2}_\wedge$.*

Proof. Consider the procedure presented in Section 2.7 and take $\Gamma^+ \supseteq \Gamma$ as a Z-maximal set, where $\Gamma \cup \{Z\} \subseteq L_\wedge$ and $\Gamma \nvdash_{\mathscr{B}_\wedge} \varphi$. By specializing property (#) for the case of $\wedge$, the completeness property to be proved is given by

$$A \wedge B \in \Gamma^+ \quad \text{iff} \quad A \in \Gamma^+ \text{ and } B \in \Gamma^+, \tag{$\wedge$}$$

for every $A, B \in L_\wedge$. From left-to-right, suppose that $A \wedge B \in \Gamma^+$. Since Γ^+ is deductively closed, $\Gamma^+ \vdash_{\mathcal{H}_\wedge} A \wedge B$. By rule c_2 and (M), $\Gamma^+, A \wedge B \vdash_{\mathcal{H}_\wedge} A$. Then, by (T), $\Gamma^+ \vdash_{\mathcal{H}_\wedge} A$, so $A \in \Gamma^+$. The proof that $B \in \Gamma^+$ is analogous using rule c_3. From right-to-left, suppose that $A \in \Gamma^+$ and $B \in \Gamma^+$, thus $\Gamma^+ \vdash_{\mathscr{B}_\wedge} A$ and $\Gamma^+ \vdash_{\mathscr{B}_\wedge} B$. By rule c_1 and (M), $\Gamma^+, A, B \vdash_{\mathscr{B}_\wedge} A \wedge B$, and hence, by (T), $\Gamma^+ \vdash_{\mathscr{B}_\wedge} A \wedge B$, then $A \wedge B \in \Gamma^+$. □

Remark 4.2.1. Notice that the relative maximality of Γ^+ was never invoked in this proof. In fact, any deductively closed set would suffice. Moreover, only the rules of $\mathscr{B}_\wedge$ were needed in the completeness proof, meaning that adding new rules to this calculus would not disprove the completeness property ($\wedge$).

The expansion $\mathcal{B}_{\wedge,\top}$ is axiomatized effortlessly by the calculus below, given Corollary 2.8.4.1.

Hilbert Calculus 6. $\mathscr{B}_{\wedge,\top}$

$$\mathscr{B}_{\wedge} \quad \mathscr{B}_{\top}$$

The expansion $\mathcal{B}_{\wedge,\bot}$ turns out to be easily axiomatized by the calculus below (an uncommon case, since expansions by the constant $\bot$ often need mixing rules):

Hilbert Calculus 7. $\mathscr{B}_{\wedge,\bot}$

$$\mathscr{B}_{\wedge} \quad \mathscr{B}_{\bot}$$

Since the completeness property $(\wedge)$ still holds in this calculus (see Remark 4.2.1), the completeness result of this expansion trivially follows because the rule b_1 implies the property $(\bot)$.

Finally, using Corollary 2.8.4.1 again, the expansion $\mathcal{B}_{\wedge,\bot,\top}$ is axiomatized by the following calculus.

Hilbert Calculus 8. $\mathscr{B}_{\wedge,\bot,\top}$

$$\mathscr{B}_{\wedge,\bot} \quad \mathscr{B}_{\top}$$

4.3 $\mathcal{B}_{\mathsf{ka}}$, $\mathcal{B}_{\mathsf{ka},\bot}$

The classical connective ka may be defined from those in $\mathcal{B}$ via the translation $\mathbf{t}(\mathsf{ka}) = \lambda p, q, r.p \wedge (q \vee r)$ and gives rise to a more complex fragment than those of the previous sections with respect to the corresponding Hilbert calculus. The rules of $\mathscr{B}_{\mathsf{ka}}$ are presented below, followed by the soundness result with respect to $\mathbb{2}_{\mathsf{ka}}$.

Hilbert Calculus 9. $\mathscr{B}_{\mathsf{ka}}$

$$\frac{A \quad B}{\mathsf{ka}(A, B, C)}\ \mathsf{ka}_1 \qquad \frac{\mathsf{ka}(A, B, B)}{B}\ \mathsf{ka}_2$$

$$\frac{\mathsf{ka}(A, B, C)}{\mathsf{ka}(A, C, B)}\ \mathsf{ka}_3 \qquad \frac{\mathsf{ka}(A, B, \mathsf{ka}(A, C, D))}{\mathsf{ka}(A, \mathsf{ka}(A, B, C), D)}\ \mathsf{ka}_4$$

$$\frac{\mathsf{ka}(A,B,C) \quad \mathsf{ka}(A,B,\mathsf{ka}(A,D,E))}{\mathsf{ka}(A,B,\mathsf{ka}(C,D,E))}\ \mathsf{ka}_5 \qquad \frac{\mathsf{ka}(A,C,\mathsf{ka}(B,D,E))}{\mathsf{ka}(A,C,B)}\ \mathsf{ka}_6$$

$$\frac{\mathsf{ka}(A,C,\mathsf{ka}(B,D,E))}{\mathsf{ka}(A,C,\mathsf{ka}(A,D,E))}\ \mathsf{ka}_7$$

Theorem 4.3.1. *The calculus $\mathscr{B}_{\mathsf{ka}}$ is sound with respect to the matrix $\mathbb{2}_{\mathsf{ka}}$.*

Proof. Let v be a $\mathbb{2}_{\mathsf{ka}}$-valuation, and, to simplify notation, denote also by v the $\mathbb{2}$-valuation v' such that $v = \mathbf{t} \circ v'$. The rule ka_1 is sound because, if $v(A) = 1$ and $v(B) = 1$, then $v(B \vee C) = 1$ and thus $v(\mathsf{ka}(A, B, C)) = 1$. For ka_2, if $v(\mathsf{ka}(A, B, B)) = 1$, then $v(B \vee B) = 1$, and $v(B) = 1$. For ka_3, if v assigns 1 to its premiss, then $v(A) = 1$ and $v(B \vee C) = 1$, and, because the interpretation of $\vee$ is commutative (see rule d_3 of the axiomatization for $\mathcal{B}_\vee$ in Section 4.5), $v(C \vee B) = 1$, and $v(\mathsf{ka}(A, C, B)) = 1$. For rule ka_4, if $v(\mathsf{ka}(A, B, \mathsf{ka}(A, C, D))) = 1$, then $v(A) = 1$ and either $v(B) = 1$ or $v(\mathsf{ka}(A, C, D)) = 1$. In the first case, $v(\mathsf{ka}(A, B, C)) = 1$, since $v(A) = 1$ too, what verifies the conclusion. In the second, either $v(C) = 1$ or $v(D) = 1$, both of which cause the conclusion to be also verified. For rule ka_5, suppose that $v(\mathsf{ka}(A, B, C)) = 1$ and $v(\mathsf{ka}(A, B, \mathsf{ka}(A, D, E))) = 1$. Then $v(A) = 1$ and either $v(B) = 1$ or $v(C) = 1$ and $v(\mathsf{ka}(A, D, E)) = 1$. In case $B = 1$, the conclusion holds because $A = 1$. Otherwise, $C = 1$ and either $D = 1$ or $E = 1$. In both those cases, the conclusion holds. For rule ka_6, if $v(\mathsf{ka}(A, C, \mathsf{ka}(B, D, E))) = 1$, then $v(A) = 1$ and either $v(C) = 1$ or $v(\mathsf{ka}(B, D, E)) = 1$. In the first case, the conclusion clearly holds. The second case implies that $v(B) = 1$, causing the conclusion to be also verified. For rule ka_7, if $v(\mathsf{ka}(A, C, \mathsf{ka}(B, D, E))) = 1$, then $v(A) = 1$ and either $v(C) = 1$ or $v(\mathsf{ka}(B, D, E)) = 1$. The first case causes the conclusion to be verified. The second one implies that either $v(D) = 1$ or $v(E) = 1$. Both cases lead to the verification of the conclusion. □

In what follows, if r is an n-ary rule, with $n \in \omega$, let r^{ka}, the ka-lifted version of r, be the rule given by the set of instances $\langle \mathsf{ka}(C, D, A_1), \ldots, \mathsf{ka}(C, D, A_n), \mathsf{ka}(C, D, B)\rangle$, where $\langle A_1, \ldots, A_n, B\rangle$ is an instance of r and $C, D \in L_{\mathsf{ka}}$. The main purpose of the next lemma is to show that all ka-lifted versions of the primitive rules of ka are derivable in $\mathscr{B}_{\mathsf{ka}}$, an important step towards the completeness of this calculus with respect to $\mathbb{2}_{\mathsf{ka}}$.

Lemma 4.3.2. *The following rules are derivable in $\mathscr{B}_{\mathsf{ka}}$:*

$$\frac{\mathsf{ka}(A,B,C)}{A}\ \mathsf{ka}_0$$

$$\frac{\mathsf{ka}(A,\mathsf{ka}(A,B,C),D)}{\mathsf{ka}(A,B,\mathsf{ka}(A,C,D))}\ \mathsf{ka}_4'$$

$$\frac{\mathsf{ka}(A, B, \mathsf{ka}(A, C, C))}{\mathsf{ka}(A, B, C)}\ \mathsf{ka}_8$$

$$\frac{\mathsf{ka}(E, D, A) \quad \mathsf{ka}(E, D, B)}{\mathsf{ka}(E, D, \mathsf{ka}(A, B, C))}\ \mathsf{ka}_1^{\mathsf{ka}}$$

$$\frac{\mathsf{ka}(D, C, \mathsf{ka}(A, B, B))}{\mathsf{ka}(D, C, B)}\ \mathsf{ka}_2^{\mathsf{ka}}$$

$$\frac{\mathsf{ka}(E, D, \mathsf{ka}(A, B, C))}{\mathsf{ka}(E, D, \mathsf{ka}(A, C, B))}\ \mathsf{ka}_3^{\mathsf{ka}}$$

$$\frac{\mathsf{ka}(F, E, \mathsf{ka}(A, B, \mathsf{ka}(A, C, D)))}{\mathsf{ka}(F, E, \mathsf{ka}(A, \mathsf{ka}(A, B, C), D))}\ \mathsf{ka}_4^{\mathsf{ka}}$$

$$\frac{\mathsf{ka}(G, F, \mathsf{ka}(A, B, C)) \quad \mathsf{ka}(G, F, \mathsf{ka}(A, B, \mathsf{ka}(A, D, E)))}{\mathsf{ka}(G, F, \mathsf{ka}(A, B, \mathsf{ka}(C, D, E)))}\ \mathsf{ka}_5^{\mathsf{ka}}$$

$$\frac{\mathsf{ka}(G, F, \mathsf{ka}(A, C, \mathsf{ka}(B, D, E)))}{\mathsf{ka}(G, F, \mathsf{ka}(A, C, B))}\ \mathsf{ka}_6^{\mathsf{ka}}$$

$$\frac{\mathsf{ka}(G, F, \mathsf{ka}(A, C, \mathsf{ka}(B, D, E)))}{\mathsf{ka}(G, F, \mathsf{ka}(A, C, \mathsf{ka}(A, D, E)))}\ \mathsf{ka}_7^{\mathsf{ka}}$$

Proof. The formally verified derivation of each rule is presented below, following what was explained in Chapter 3.

- ka_0

```
theorem ka₀ {a b c : Prop} (h₁ : ka a b c) : a :=
  have h₃ : ka (ka a b c) (ka a b c) a, from ka₁ h₁ h₁,
  have h₄ : ka (ka a b c) a (ka a b c), from ka₃ h₃,
  have h₅ : ka (ka a b c) a a, from ka₆ h₄,
  show a, from ka₂ h₅
```

- ka_4'

```
theorem ka₄' {a b c d : Prop} (h₁ : ka d (ka d c a) b) : ka d c (ka d a b) :=
  have h₂ : ka d b (ka d c a), from ka₃ h₁,
  have h₃ : ka d (ka d b c) a, from ka₄ h₂,
  have h₄ : ka d a (ka d b c), from ka₃ h₃,
  have h₅ : ka d (ka d a b) c, from ka₄ h₄,
  show ka d c (ka d a b), from ka₃ h₅
```

- ka_8

```
theorem ka₈ {a b c : Prop} (h₁ : ka a b (ka a c c)) : ka a b c :=
  have h₂ : ka a (ka a b c) c, from ka₄ h₁,
  have h₃ : ka a c (ka a b c), from ka₃ h₂,
  have h₄ : a, from ka₀ h₂,
  have h₅ : ka a (ka a c (ka a b c)) b, from ka₁ h₄ h₃,
  have h₆ : ka a b (ka a c (ka a b c)), from ka₃ h₅,
  have h₇ : ka a (ka a b c) (ka a b c), from ka₄ h₆,
  show ka a b c, from ka₂ h₇
```

- ka_1^{ka}

```
theorem ka₁_ka {a b c d e : Prop} (h₁ : ka e d a) (h₂ : ka e d b) : ka e d (ka a b c) :=
  have h₃ : e, from ka₀ h₂,
  have h₄ : ka e (ka e d b) c, from ka₁ h₃ h₂,
  have h₅ : ka e d (ka e b c), from ka₄' h₄,
  show ka e d (ka a b c), from ka₅ h₁ h₅
```

- ka_2^{ka}

```
theorem ka₂_ka {a b c d : Prop} (h₁ : ka d c (ka a b b)) : ka d c b :=
  have h₂ : ka d c (ka d b b), from ka₇ h₁,
  show ka d c b, from ka₈ h₂
```

- ka_3^{ka}

```
theorem ka₃_ka {a b c d e : Prop} (h₁ : ka e d (ka a b c)) : ka e d (ka a c b) :=
  have h₂ : e, from ka₀ h₁,
  have h₃ : ka e d a, from ka₆ h₁,
  have h₄ : ka e d (ka e b c), from ka₇ h₁,
  have h₅ : ka e (ka e d b) c, from ka₄ h₄,
  have h₆ : ka e (ka e (ka e d b) c) b, from ka₁ h₂ h₅,
  have h₇ : ka e (ka e d b) (ka e c b), from ka₄' h₆,
  have h₈ : ka e d (ka e b (ka e c b)), from ka₄' h₇,
  have h₉ : ka e (ka e b (ka e c b)) d, from ka₃ h₈,
  have h₁₀ : ka e b (ka e (ka e c b) d), from ka₄' h₉,
  have h₁₁ : ka e (ka e b (ka e (ka e c b) d)) c, from ka₁ h₂ h₁₀,
  have h₁₂ : ka e c (ka e b (ka e (ka e c b) d)), from ka₃ h₁₁,
  have h₁₃ : ka e (ka e c b) (ka e (ka e c b) d), from ka₄ h₁₂,
  have h₁₄ : ka e (ka e (ka e c b) (ka e c b)) d, from ka₄ h₁₃,
  have h₁₅ : ka e d (ka e (ka e c b) (ka e c b)), from ka₃ h₁₄,
```

```
  have h₁₆ : ka e d (ka e c b), from ka₂_ka h₁₅,
  show ka e d (ka a c b), from ka₅ h₃ h₁₆
```

- $\mathsf{ka}_4^{\mathsf{ka}}$

```
theorem ka₄_ka {a b c d e f : Prop} (h₁ : ka f e (ka a b (ka a c d))) :
  ka f e (ka a (ka a b c) d):=
  have h₂ : ka f e (ka f b (ka a c d)), from ka₇ h₁,
  have h₃ : ka f (ka f e b) (ka a c d), from ka₄ h₂,
  have h₄ : ka f (ka f e b) (ka f c d), from ka₇ h₃,
  have h₅ : ka f (ka f e b) (ka f d c), from ka₃_ka h₄,
  have h₆ : ka f (ka f (ka f e b) d) c, from ka₄ h₅,
  have h₇ : f, from ka₀ h₁,
  have h₈ : ka f (ka f (ka f (ka f e b) d) c) b, from ka₁ h₇ h₆,
  have h₉ : ka f (ka f (ka f e b) d) (ka f c b), from ka₄' h₈,
  have h₁₀ : ka f (ka f (ka f e b) d) (ka f b c), from ka₃_ka h₉,
  have h₁₁ : ka f (ka f e b) (ka f d (ka f b c)), from ka₄' h₁₀,
  have h₁₂ : ka f (ka f e b) (ka f (ka f b c) d), from ka₃_ka h₁₁,
  let g := ka f (ka f b c) d in
    have h₁₃ : ka f (ka f e b) g, from h₁₂,
    have h₁₄ : ka f e (ka f b g), from ka₄' h₁₃,
    have h₁₅ : ka f e (ka f g b), from ka₃_ka h₁₄,
    have h₁₆ : ka f (ka f e g) b, from ka₄ h₁₅,
    have h₁₇ : ka f (ka f (ka f e g) b) c, from ka₁ h₇ h₁₆,
    have h₁₈ : ka f (ka f e g) (ka f b c), from ka₄' h₁₇,
    have h₁₉ : ka f (ka f (ka f e g) (ka f b c)) d, from ka₁ h₇ h₁₈,
    have h₂₀ : ka f (ka f e g) (ka f (ka f b c) d), from ka₄' h₁₉,
    have h₂₁ : ka f (ka f e g) g, from h₂₀,
    have h₂₂ : ka f e (ka f g g), from ka₄' h₂₁,
    have h₂₃ : ka f e g, from ka₂_ka h₂₂,
    have h₂₄ : ka f e (ka f (ka f b c) d), from h₂₃,
    have h₂₅ : ka f e a, from ka₆ h₁,
    have h₂₆ : ka f (ka f e a) d, from ka₁ h₇ h₂₅,
    have h₂₇ : ka f e (ka f a d), from ka₄' h₂₆,
    have h₂₈ : ka f e (ka f d a), from ka₃_ka h₂₇,
    have h₂₉ : ka f (ka f e d) a, from ka₄ h₂₈,
    have h₃₀ : ka f e (ka f d (ka f b c)), from ka₃_ka h₂₄,
    have h₃₁ : ka f (ka f e d) (ka f b c), from ka₄ h₃₀,
    have h₃₂ : ka f (ka f e d) (ka a b c), from ka₅ h₂₉ h₃₁,
    have h₃₃ : ka f e (ka f d (ka a b c)), from ka₄' h₃₂,
    have h₃₄ : ka f e (ka a d (ka a b c)), from ka₅ h₂₅ h₃₃,
    show ka f e (ka a (ka a b c) d), from ka₃_ka h₃₄
```

- ka_5^{ka}

```
theorem ka₅_ka {a b c d e f g : Prop}
        (h₁ : ka g f (ka a b c))
        (h₂ : ka g f (ka a b (ka a d e))) :
        ka g f (ka a b (ka c d e)) :=
  have h₃ : ka g f (ka g b c), from ka₇ h₁,
  have h₄ : ka g (ka g f b) c, from ka₄ h₃,
  have h₅ : ka g f (ka g b (ka a d e)), from ka₇ h₂,
  have h₆ : ka g (ka g f b) (ka a d e), from ka₄ h₅,
  have h₇ : ka g (ka g f b) (ka g d e), from ka₇ h₆,
  have h₈ : ka g (ka g f b) (ka c d e), from ka₅ h₄ h₇,
  have h₉ : ka g f (ka g b (ka c d e)), from ka₄' h₈,
  have h₁₀ : ka g f a, from ka₆ h₁,
  show ka g f (ka a b (ka c d e)), from ka₅ h₁₀ h₉
```

- ka_6^{ka}

```
theorem ka₆_ka {a b c d e f g : Prop} (h₁ : ka g f (ka a c (ka b d e))) :
  ka g f (ka a c b) :=
  have h₂ : ka g f (ka g c (ka b d e)), from ka₇ h₁,
  have h₃ : ka g (ka g f c) (ka b d e), from ka₄ h₂,
  have h₄ : ka g (ka g f c) b, from ka₆ h₃,
  have h₅ : ka g f (ka g c b), from ka₄' h₄,
  have h₆ : ka g f a, from ka₆ h₁,
  show ka g f (ka a c b), from ka₅ h₆ h₅
```

- ka_7^{ka}

```
theorem ka₇_ka {a b c d e f g : Prop} (h₁ : ka g f (ka a c (ka b d e))) :
  ka g f (ka a c (ka a d e)) :=
  have h₂ : ka g f a, from ka₆ h₁,
  have h₃ : g, from ka₀ h₁,
  have h₄ : ka g (ka g f a) c, from ka₁ h₃ h₂,
  have h₅ : ka g f (ka g a c), from ka₄' h₄,
  have h₆ : ka g f (ka g c a), from ka₃_ka h₅,
  have h₇ : ka g (ka g f c) a, from ka₄ h₆,
  have h₈ : ka g f (ka g c (ka b d e)), from ka₇ h₁,
  have h₉ : ka g (ka g f c) (ka b d e), from ka₄ h₈,
  have h₁₀ : ka g (ka g f c) (ka g d e), from ka₇ h₉,
```

```
have h11 : ka g (ka g f c) (ka a d e), from ka5 h7 h10,
have h12 : ka g f (ka g c (ka a d e)), from ka4' h11,
show ka g f (ka a c (ka a d e)), from ka5 h2 h12
```

□

Next, we present the monotonicity property m_{ka} and the deduction theorem δ_{ka}, and prove that they hold in $\mathscr{B}_{\mathsf{ka}}$, culminating in the completeness of this calculus.

Lemma 4.3.3. *The following property holds for $\vdash_{\mathscr{B}_{\mathsf{ka}}}$:*

$$\text{for all } \Gamma \cup \{A, B, C, D\} \subseteq L_{\mathsf{ka}}. \text{ if } \Gamma, A \vdash_{\mathscr{B}_{\mathsf{ka}}} B \text{ then } \Gamma, \mathsf{ka}(C, D, A) \vdash_{\mathscr{B}_{\mathsf{ka}}} \mathsf{ka}(C, D, B) \quad (m_{\mathsf{ka}})$$

Proof. Let $\Gamma \cup \{A, B, C, D\} \subseteq L_{\mathsf{ka}}$. Suppose that $\Gamma, A \vdash_{\mathscr{B}_{\mathsf{ka}}} B$ and that this is witnessed by the derivation $P_1, \ldots, P_n$, where $P_n = B$, for some $n \in \omega$. We will prove by induction on this derivation that $\Gamma, \mathsf{ka}(C, D, A) \vdash_{\mathscr{B}_{\mathsf{ka}}} \mathsf{ka}(C, D, P_j)$, for all $1 \leq j \leq n$. In the base case, where $j = 1$, P_1 is either equal to A or is in Γ, since no axioms are available in the system. In the first case, using that $P_1 = A$, together with (R) and (M), we get $\Gamma, \mathsf{ka}(C, D, A) \vdash_{\mathscr{B}_{\mathsf{ka}}} \mathsf{ka}(C, D, P_1)$. In the second case, if $P_1 \in \Gamma$, then, by taking the context $\Gamma' := \Gamma \cup \{\mathsf{ka}(C, D, A)\}$, the following proves $\mathsf{ka}(C, D, P_1)$:

(1)	P_1	$P_1 \in \Gamma'$
(2)	$\mathsf{ka}(C, D, A)$	$\mathsf{ka}(C, D, A) \in \Gamma'$
(3)	C	2 ka_0
(4)	$\mathsf{ka}(C, P_1, D)$	3, 1 ka_1
(5)	$\mathsf{ka}(C, D, P_1)$	4 ka_3

For the inductive step, suppose that $\Gamma, \mathsf{ka}(C, D, A) \vdash_{\mathscr{B}_{\mathsf{ka}}} \mathsf{ka}(C, D, P_k)$, for all $k < j$, with $j > 1$. Then, P_j is either A, is in Γ or results from the application of the instance $\langle P_{k_1}, \ldots, P_{k_m}, P_j \rangle$ of one of the m-ary primitive rules, say ka_s for $1 \leq \mathsf{s} \leq 7$, to the premisses $P_{k_1}, \ldots, P_{k_m}$, where $k_l < j$, for all $1 \leq l \leq m$. The first two cases goes like in the base case. For the third case, it is true that $\mathsf{ka}(C, D, P_{k_1}), \ldots, \mathsf{ka}(C, D, P_{k_m})$ are provable from $\Gamma, \mathsf{ka}(C, D, A)$ by the inductive hypothesis. Then, by applying the corresponding rule $\mathsf{ka}_\mathsf{s}^{\mathsf{ka}}$, proved to be derivable in Lemma 4.3.2, using those formulas as premisses, we derive $\mathsf{ka}(C, D, P_j)$, as desired. The proof of the present lemma is precisely the case $j = n$. □

Lemma 4.3.4. *The following property holds for* $\vdash_{\mathscr{B}_{\mathsf{ka}}}$*:*

$$\text{for all } \Gamma \cup \{A, B, C, D\} \subseteq L_{\mathsf{ka}}.\text{ if } \Gamma, A \vdash_{\mathscr{B}_{\mathsf{ka}}} C \text{ and } \Gamma, B \vdash_{\mathscr{B}_{\mathsf{ka}}} C \text{ then } \Gamma, \mathsf{ka}(D, A, B) \vdash_{\mathscr{B}_{\mathsf{ka}}} C \qquad (\delta_{\mathsf{ka}})$$

Proof. Let $\Gamma \cup \{A, B, C, D\} \subseteq L_{\mathsf{ka}}$ and suppose that $\Gamma, A \vdash_{\mathscr{B}_{\mathsf{ka}}} C$ and $\Gamma, B \vdash_{\mathscr{B}_{\mathsf{ka}}} C$. Then, the following reasoning proves the present lemma:

(1)	$\Gamma, A \vdash_{\mathscr{B}_{\mathsf{ka}}} C$	Assumption
(2)	$\Gamma, B \vdash_{\mathscr{B}_{\mathsf{ka}}} C$	Assumption
(3)	$\Gamma, \mathsf{ka}(D, C, A) \vdash_{\mathscr{B}_{\mathsf{ka}}} \mathsf{ka}(D, C, C)$	1 m_{ka}
(4)	$\mathsf{ka}(D, C, C) \vdash_{\mathscr{B}_{\mathsf{ka}}} C$	ka_2
(5)	$\Gamma, \mathsf{ka}(D, C, A), \mathsf{ka}(D, C, C) \vdash_{\mathscr{B}_{\mathsf{ka}}} C$	4 (M)
(6)	$\Gamma, \mathsf{ka}(D, C, A) \vdash_{\mathscr{B}_{\mathsf{ka}}} C$	3, 5 (T)
(7)	$\Gamma, \mathsf{ka}(D, A, B) \vdash_{\mathscr{B}_{\mathsf{ka}}} \mathsf{ka}(D, A, C)$	2 m_{ka}
(8)	$\mathsf{ka}(D, A, C) \vdash_{\mathscr{B}_{\mathsf{ka}}} \mathsf{ka}(D, C, A)$	ka_3
(9)	$\Gamma, \mathsf{ka}(D, A, B), \mathsf{ka}(D, A, C) \vdash_{\mathscr{B}_{\mathsf{ka}}} \mathsf{ka}(D, C, A)$	8 (M)
(10)	$\Gamma, \mathsf{ka}(D, A, B) \vdash_{\mathscr{B}_{\mathsf{ka}}} \mathsf{ka}(D, C, A)$	7, 9 (T)
(11)	$\Gamma, \mathsf{ka}(D, A, B), \mathsf{ka}(D, C, A) \vdash_{\mathscr{B}_{\mathsf{ka}}} C$	6 (M)
(12)	$\Gamma, \mathsf{ka}(D, A, B) \vdash_{\mathscr{B}_{\mathsf{ka}}} C$	10, 11 (T)

□

Theorem 4.3.5. *The calculus* $\mathscr{B}_{\mathsf{ka}}$ *is complete with respect to the matrix* $\mathfrak{Q}_{\mathsf{ka}}$*.*

Proof. Following the procedure presented in Section 2.7, let $\Gamma \cup \{Z\} \subseteq L_{\mathsf{ka}}$ and take the Z-maximal theory $\Gamma^+ \supseteq \Gamma$ via the Lindenbaum-Asser Lemma. From the interpretation of ka in $\mathfrak{Q}_{\mathsf{ka}}$ and property (#), the completeness property (ka) is given by:

$$\mathsf{ka}(A, B, C) \in \Gamma^+ \text{ iff } A \in \Gamma^+ \text{ and } (B \in \Gamma^+ \text{ or } C \in \Gamma^+) \qquad (\mathsf{ka})$$

From the right to the left, suppose that $A \in \Gamma^+$ and $B \in \Gamma^+$, thus $\Gamma^+ \vdash_{\mathscr{B}_{\mathsf{ka}}} A$ and $\Gamma^+ \vdash_{\mathscr{B}_{\mathsf{ka}}} B$. The instance of ka_1 given by $\langle A, B, \mathsf{ka}(A, B, C)\rangle$ alongside with (M) guarantee that $\Gamma^+, A, B \vdash_{\mathscr{B}_{\mathsf{ka}}} \mathsf{ka}(A, B, C)$. Using the latter with the previous consecutions, by an appeal to (T), produces $\Gamma^+ \vdash_{\mathscr{B}_{\mathsf{ka}}} \mathsf{ka}(A, B, C)$, thus $\mathsf{ka}(A, B, C) \in \Gamma^+$. The proof in the case $A \in \Gamma^+$ and $C \in \Gamma^+$ is analogous.

From the left to the right, suppose that (a): $\mathsf{ka}(A, B, C) \in \Gamma^+$. Then, $\Gamma^+ \vdash_{\mathscr{B}_{\mathsf{ka}}} \mathsf{ka}(A, B, C)$. By rule ka_0 and (M), $\Gamma^+, \mathsf{ka}(A, B, C) \vdash_{\mathscr{B}_{\mathsf{ka}}} A$, which, together with the previous consecution, yields $\Gamma^+ \vdash_{\mathscr{B}_{\mathsf{ka}}} \mathrm{A}$ by (T). Now, we have to show that $\mathrm{B} \in \Gamma^+$ or $\mathrm{C} \in \Gamma^+$. Let us work by contradiction: assume that $\mathrm{B}, \mathrm{C} \notin \Gamma^+$, then, by Corollary 2.6.1.1, $\Gamma^+, \mathrm{B} \vdash_{\mathscr{B}_{\mathsf{ka}}} \mathrm{Z}$ and $\Gamma^+, \mathrm{C} \vdash_{\mathscr{B}_{\mathsf{ka}}} \mathrm{Z}$. Hence, by δ_{ka} (see Lemma 4.3.4), $\Gamma^+, \mathsf{ka}(\mathrm{A}, \mathrm{B}, \mathrm{C}) \vdash_{\mathscr{B}_{\mathsf{ka}}} \mathrm{Z}$, yielding, together with (a), $\Gamma^+ \vdash_{\mathscr{B}_{\mathsf{ka}}} \mathrm{Z}$ by (T), an absurd. □

Remark 4.3.1. Notice that if a new rule r is added to $\mathscr{B}_{\mathsf{ka}}$, then deriving its ka-lifted version, namely r^{ka}, causes the completeness property (ka) to be preserved in the expanded calculus.

The calculus for the expansion $\mathcal{B}_{\mathsf{ka},\bot}$ comes from the calculus for $\mathcal{B}_{\mathsf{ka}}$ by adding a new rule of interaction, as presented below:

Hilbert Calculus 10. $\mathscr{B}_{\mathsf{ka},\bot}$

$$\mathscr{B}_{\mathsf{ka}} \qquad \frac{\mathsf{ka}(\mathrm{A}, \mathrm{B}, \bot)}{\mathsf{ka}(\mathrm{A}, \mathrm{B}, \mathrm{C})}\ \mathsf{kab}_1$$

Theorem 4.3.6. *The calculus $\mathscr{B}_{\mathsf{ka},\bot}$ is sound with respect to the matrix $\mathbb{2}_{\mathsf{ka},\bot}$.*

Proof. Remember that we already proved the soundness of $\mathscr{B}_{\mathsf{ka}}$ (see Theorem 4.3.1), so it remains to show that the new rule is sound. For that, let v be an arbitrary $\mathbb{2}_{\mathsf{ka},\bot}$-valuation and suppose that $v(\mathsf{ka}(\mathrm{A}, \mathrm{B}, \bot)) = 1$, thus, from the truth-table of ka, $v(\mathrm{A}) = 1$ and either $v(\mathrm{B}) = 1$ or $v(\bot) = 1$. Since this second case is impossible, we necessarily have $v(\mathrm{B}) = 1$, and the targeted conclusion follows. □

The following lemma deals with the derivability of the rules necessary for preserving the completeness properties of both connectives of $\mathcal{B}_{\mathsf{ka},\bot}$.

Lemma 4.3.7. *The following rules are derivable in $\mathscr{B}_{\mathsf{ka},\bot}$:*

$$\frac{\mathsf{ka}(\mathrm{D}, \mathrm{E}, \mathsf{ka}(\mathrm{A}, \mathrm{B}, \bot))}{\mathsf{ka}(\mathrm{D}, \mathrm{E}, \mathsf{ka}(\mathrm{A}, \mathrm{B}, \mathrm{C}))}\ \mathsf{kab}_1^{\mathsf{ka}}$$

$$\frac{\bot}{\mathrm{A}}\ \mathsf{b}_1$$

Proof. The formally verified derivation of each rule is presented below:

- $\mathsf{kab}_1^{\mathsf{ka}}$

```
theorem kab₁_ka {a b c d e : Prop} (h₁ : ka d e (ka a b bot)) : ka d e (ka a b c) :=
  have h₂ : ka d e (ka d b bot), from ka.ka₇ h₁,
  have h₃ : ka d (ka d e b) bot, from ka.ka₄ h₂,
  have h₄ : ka d (ka d e b) c, from kab₁ h₃,
  have h₅ : ka d e a, from ka.ka₆ h₁,
  have h₆ : ka d e (ka d b c), from ka.ka₄' h₄,
  show ka d e (ka a b c), from ka.ka₅ h₅ h₆
```

- b_1

```
theorem b₁ {a : Prop} (h₁ : bot) : a :=
  have h₂ : ka bot bot bot, from ka.ka₁ h₁ h₁,
  have h₃ : ka bot bot a, from kab₁ h₂,
  have h₄ : ka bot a bot, from ka.ka₃ h₃,
  have h₅ : ka bot a a, from kab₁ h₄,
  show a, from ka.ka₂ h₅
```

□

Theorem 4.3.8. *The calculus $\mathscr{B}_{\mathsf{ka},\perp}$ is complete with respect to the matrix $\mathfrak{2}_{\mathsf{ka},\perp}$.*

Proof. According to what was pointed out in Remark 4.1.1 and Remark 4.3.1, the derivability of b_1 and $\mathsf{kab}_1^{\mathsf{ka}}$ imply the completeness properties $(\perp)$ and (ka), respectively. □

4.4 $\mathcal{B}_{\mathsf{ki}}$, $\mathcal{B}_{\mathsf{ki},\perp}$

The classical connective ki may be defined from those in $\mathcal{B}$ by means of the translation $\mathbf{t}(\mathsf{ki}) = \lambda p, q, r.p \wedge (q \to r)$. The proposed calculus for the fragment $\mathcal{B}_{\mathsf{ki}}$, with nine rules and no axioms, is presented below, followed immediately by the soundness proof.

Hilbert Calculus 11. $\mathscr{B}_{\mathsf{ki}}$

$$\frac{B \quad \mathsf{ki}(A, B, C)}{C}\ \mathsf{ki}_1$$

$$\frac{A}{\mathsf{ki}(A, B, \mathsf{ki}(A, C, B))}\ \mathsf{ki}_2$$

$$\frac{\mathsf{ki}(\mathrm{B},\mathrm{F},\mathrm{A})}{\mathsf{ki}(\mathrm{B},\mathrm{F},\mathsf{ki}(\mathrm{A},\mathsf{ki}(\mathrm{A},\mathrm{C},\mathsf{ki}(\mathrm{A},\mathrm{D},\mathrm{E})),\mathsf{ki}(\mathrm{A},\mathsf{ki}(\mathrm{A},\mathrm{C},\mathrm{D}),\mathsf{ki}(\mathrm{A},\mathrm{C},\mathrm{E}))))}\ \mathsf{ki}_3$$

$$\frac{\mathsf{ki}(\mathrm{B},\mathrm{E},\mathrm{A})}{\mathsf{ki}(\mathrm{B},\mathrm{E},\mathsf{ki}(\mathrm{A},\mathsf{ki}(\mathrm{A},\mathsf{ki}(\mathrm{A},\mathrm{C},\mathrm{D}),\mathrm{C}),\mathrm{C}))}\ \mathsf{ki}_4$$

$$\frac{\mathsf{ki}(\mathrm{A},\mathrm{B},\mathsf{ki}(\mathrm{A},\mathrm{C},\mathrm{D}))}{\mathsf{ki}(\mathrm{A},\mathsf{ki}(\mathrm{B},\mathrm{B},\mathrm{C}),\mathrm{D})}\ \mathsf{ki}_5$$

$$\frac{\mathsf{ki}(\mathrm{A},\mathsf{ki}(\mathrm{B},\mathrm{B},\mathrm{C}),\mathrm{D})}{\mathsf{ki}(\mathrm{A},\mathrm{B},\mathsf{ki}(\mathrm{A},\mathrm{C},\mathrm{D}))}\ \mathsf{ki}_6$$

$$\frac{\mathsf{ki}(\mathrm{A},\mathrm{E},\mathrm{B})\quad \mathsf{ki}(\mathrm{A},\mathrm{E},\mathsf{ki}(\mathrm{A},\mathrm{C},\mathrm{D}))}{\mathsf{ki}(\mathrm{A},\mathrm{E},\mathsf{ki}(\mathrm{B},\mathrm{C},\mathrm{D}))}\ \mathsf{ki}_7$$

$$\frac{\mathsf{ki}(\mathrm{A},\mathrm{E},\mathsf{ki}(\mathrm{B},\mathrm{C},\mathrm{D}))}{\mathsf{ki}(\mathrm{A},\mathrm{E},\mathrm{B})}\ \mathsf{ki}_8$$

$$\frac{\mathsf{ki}(\mathrm{A},\mathrm{E},\mathsf{ki}(\mathrm{B},\mathrm{C},\mathrm{D}))}{\mathsf{ki}(\mathrm{A},\mathrm{E},\mathsf{ki}(\mathrm{A},\mathrm{C},\mathrm{D}))}\ \mathsf{ki}_9$$

Theorem 4.4.1. *The calculus $\mathscr{B}_{\mathsf{ki}}$ is sound with respect to the matrix $\mathbb{2}_{\mathsf{ki}}$.*

Proof. Let v be a $\mathbb{2}_{\mathsf{ki}}$-valuation, and, to simplify notation, denote also by v the $\mathbb{2}$-valuation v' such that $v = \mathbf{t} \circ v'$. For ki_1, suppose that $v(\mathrm{B}) = 1$ and $v(\mathsf{ki}(\mathrm{A},\mathrm{B},\mathrm{C})) = 1$. Then $v(\mathrm{A}) = 1$ for sure, and $v(\mathrm{C}) = 1$ necessarily, otherwise $v(\mathrm{B} \to \mathrm{C}) = 0$, causing $v(\mathsf{ki}(\mathrm{A},\mathrm{B},\mathrm{C})) = 0$, a contradiction. For ki_2, suppose that $v(\mathsf{ki}(\mathrm{A},\mathrm{B},\mathsf{ki}(\mathrm{A},\mathrm{C},\mathrm{B}))) = 0$. Then we have two cases, either $v(\mathrm{A}) = 0$ or $v(\mathrm{B} \to \mathsf{ki}(\mathrm{A},\mathrm{C},\mathrm{B})) = 0$. The latter implies that $v(\mathrm{B}) = 1$ and $v(\mathsf{ki}(\mathrm{A},\mathrm{C},\mathrm{B})) = 0$. Since $v(\mathrm{C} \to \mathrm{B}) = 1$ because $v(\mathrm{B}) = 1$, we have $v(\mathrm{A}) = 0$. For ki_3, we have two cases to consider, under the assignments that make $v(\mathsf{ki}(\mathrm{B},\mathrm{F},\mathrm{A})) = 1$: either $v(\mathrm{F}) = 0$ or $v(\mathrm{F}) = 1$. In the first case, we have the conclusion being falsified by v, because F is the antecedent of the outermost implication that constitutes the conclusion. The remaining case is $v(\mathrm{B}) = 1, v(\mathrm{F}) = 1$ and $v(\mathrm{A}) = 1$. Notice that if $v(\mathrm{C}) = 0$, the conclusion also takes the value 1. Now, if $v(\mathrm{C}) = 1$, taking $v(\mathrm{D}) = 0$ also makes the conclusion to receive the value 1. If $v(\mathrm{D}) = 1$ and taking $v(\mathrm{E}) = 0$, we have $v(\mathsf{ki}(\mathrm{A},\mathrm{C},\mathsf{ki}(\mathrm{A},\mathrm{D},\mathrm{E}))) = 0$, validating the conclusion, and if $v(\mathrm{E}) = 1$ then trivially the conclusion is validated. For ki_4, if the conclusion is evaluated to 0, then either $v(\mathrm{B}) = 0$, falsifying the premiss, or $v(\mathrm{E}) = 1$ and $v(\mathsf{ki}(\mathrm{A},\mathsf{ki}(\mathrm{A},\mathsf{ki}(\mathrm{A},\mathrm{C},\mathrm{D}),\mathrm{C}),\mathrm{C})) = 0$. In this case, either $v(\mathrm{A}) = 0$ or $v(\mathsf{ki}(\mathrm{A},\mathrm{C},\mathrm{D}) \to \mathrm{C}) = 1$ and $v(\mathrm{C}) = 0$. This last case makes $v(\mathsf{ki}(\mathrm{A},\mathrm{C},\mathrm{D})) = 0$ and $v(\mathrm{C} \to \mathrm{D}) = 1$, so $v(\mathrm{A}) = 0$, falsifying the premiss. For ki_5, if the conclusion is evaluated to 0, then either $v(\mathrm{A}) = 0$, falsifying the premiss, or $v(\mathsf{ki}(\mathrm{B},\mathrm{B},\mathrm{C}) \to \mathrm{D}) = 0$. In this case, $v(\mathsf{ki}(\mathrm{B},\mathrm{B},\mathrm{C})) = 1$ and $v(\mathrm{D}) = 0$. Then, we have $v(\mathrm{B}) = 1$ and $v(\mathrm{C}) = 1$, making the premiss false, because $v(\mathrm{C} \to \mathrm{D}) = 0$. For ki_6, if the conclusion is evaluated to 0, then either $v(\mathrm{A}) = 0$ or $v(\mathrm{B} \to \mathsf{ki}(\mathrm{A}, \mathrm{C}, \mathrm{D})) = 0$. In this case, $v(\mathrm{B}) = 1$ and $v(\mathsf{ki}(\mathrm{A}, \mathrm{C}, \mathrm{D})) = 0$. But now, assuming $v(\mathrm{A}) = 1$, we have $v(\mathrm{C} \to \mathrm{D}) = 0$, so $v(\mathrm{C}) = 1$ and $v(\mathrm{D}) = 0$, causing $v(\mathsf{ki}(\mathrm{B}, \mathrm{B}, \mathrm{C}) \to \mathrm{D}) = 0$, falsifying the premiss. For ki_7, suppose that the conclusion is evaluated to 0. Then either $v(\mathrm{A}) = 0$ or $v(\mathrm{E} \to \mathsf{ki}(\mathrm{B}, \mathrm{C}, \mathrm{D})) = 0$. In the latter, taking $v(\mathrm{A}) = 1$, we have $v(\mathrm{E}) = 1$ and $v(\mathsf{ki}(\mathrm{B}, \mathrm{C}, \mathrm{D})) = 0$, but then $v(\mathrm{B}) = 1$ and $v(\mathrm{C} \to \mathrm{D}) = 0$ and the premiss $\mathsf{ki}(\mathrm{A}, \mathrm{E}, \mathsf{ki}(\mathrm{A}, \mathrm{C}, \mathrm{D}))$ is evaluated to 0. For ki_8, suppose that the conclusion is evaluated to 0. Then either $v(\mathrm{A}) = 0$ or $v(\mathrm{E} \to \mathrm{B}) = 0$. In this case, $v(\mathrm{E}) = 1$ and $v(\mathrm{B}) = 0$, causing $v(\mathsf{ki}(\mathrm{B}, \mathrm{C}, \mathrm{D})) = 0$, falsifying the entire premiss. For ki_9, if the conclusion is evaluated to 0, then either $v(\mathrm{A}) = 0$ or $v(\mathrm{E} \to \mathsf{ki}(\mathrm{A}, \mathrm{C}, \mathrm{D})) = 0$. In this case, $v(\mathrm{E}) = 1$ and $v(\mathsf{ki}(\mathrm{A}, \mathrm{C}, \mathrm{D})) = 0$, thus, since $v(\mathrm{A}) = 1$, $v(\mathrm{C} \to \mathrm{D}) = 0$, meaning that the premiss is falsified no matter the assignment $v(\mathrm{B})$. □

In what follows, if r is an n-ary rule, with $n \in \omega$, consider r^{ki}, the ki-lifted version of r, the rule given by the set of instances $\langle \mathsf{ki}(\mathrm{C}, \mathrm{D}, \mathrm{A}_1), \ldots, \mathsf{ki}(\mathrm{C}, \mathrm{D}, \mathrm{A}_n), \mathsf{ki}(\mathrm{C}, \mathrm{D}, \mathrm{B}) \rangle$, where $\langle \mathrm{A}_1, \ldots, \mathrm{A}_n, \mathrm{B} \rangle$ is an instance of r and $\mathrm{C}, \mathrm{D} \in L_{\mathsf{ki}}$. The main purpose of the next lemma is to show that $\mathscr{B}_{\mathsf{ki}}$ has the ki-lifted versions of its primitive rules as derivable rules, an important fact for proving completeness with respect to $\mathfrak{2}_{\mathsf{ki}}$.

Lemma 4.4.2. *The following rules are derivable in $\mathscr{B}_{\mathsf{ki}}$:*

$$\frac{\mathrm{A} \quad \mathrm{C}}{\mathsf{ki}(\mathrm{A}, \mathrm{B}, \mathrm{C})}\ \mathsf{ki}_{10}$$

$$\frac{\mathrm{A}}{\mathsf{ki}(\mathrm{A}, \mathsf{ki}(\mathrm{A}, \mathrm{B}, \mathsf{ki}(\mathrm{A}, \mathrm{C}, \mathrm{D})), \mathsf{ki}(\mathrm{A}, \mathsf{ki}(\mathrm{A}, \mathrm{B}, \mathrm{C}), \mathsf{ki}(\mathrm{A}, \mathrm{B}, \mathrm{D})))}\ \mathsf{ki}'_3$$

$$\frac{\mathrm{A}}{\mathsf{ki}(\mathrm{A}, \mathsf{ki}(\mathrm{A}, \mathsf{ki}(\mathrm{A}, \mathrm{C}, \mathrm{B}), \mathrm{C}), \mathrm{C})}\ \mathsf{ki}'_4$$

$$\frac{\mathsf{ki}(\mathrm{A}, \mathrm{B}, \mathrm{C})}{\mathrm{A}}\ \mathsf{ki}_0$$

$$\frac{\mathrm{A}}{\mathsf{ki}(\mathrm{A}, \mathrm{B}, \mathrm{B})}\ \mathsf{ki}_{11}$$

$$\frac{\mathsf{ki}(\mathrm{A}, \mathrm{B}, \mathsf{ki}(\mathrm{A}, \mathrm{C}, \mathrm{D}))}{\mathsf{ki}(\mathrm{A}, \mathrm{C}, \mathsf{ki}(\mathrm{A}, \mathrm{B}, \mathrm{D}))}\ \mathsf{ki}_{12}$$

$$\frac{\mathsf{ki}(\mathrm{A}, \mathrm{B}, \mathsf{ki}(\mathrm{A}, \mathrm{B}, \mathrm{C}))}{\mathsf{ki}(\mathrm{A}, \mathrm{B}, \mathrm{C})}\ \mathsf{ki}_{13}$$

$$\frac{\mathsf{ki}(\mathrm{A}, \mathrm{B}, \mathrm{D})}{\mathsf{ki}(\mathrm{B}, \mathrm{B}, \mathsf{ki}(\mathrm{A}, \mathrm{C}, \mathrm{D}))}\ \mathsf{ki}_{14}$$

$$\frac{\mathsf{ki}(\mathrm{A}, \mathrm{B}, \mathrm{D})}{\mathsf{ki}(\mathrm{C}, \mathrm{C}, \mathsf{ki}(\mathrm{A}, \mathrm{B}, \mathrm{D}))}\ \mathsf{ki}_{15}$$

$$\frac{\mathsf{ki}(A,B,C) \quad \mathsf{ki}(A,C,D)}{\mathsf{ki}(A,B,D)}\ \mathsf{ki}_{16}$$

$$\frac{\mathsf{ki}(D,E,B) \quad \mathsf{ki}(D,E,\mathsf{ki}(A,B,C))}{\mathsf{ki}(D,E,C)}\ \mathsf{ki}_{1}^{\mathsf{ki}}$$

$$\frac{\mathsf{ki}(D,E,A)}{\mathsf{ki}(D,E,\mathsf{ki}(A,B,\mathsf{ki}(A,C,B)))}\ \mathsf{ki}_{2}^{\mathsf{ki}}$$

$$\frac{\mathsf{ki}(D,E,A) \quad \mathsf{ki}(D,E,C)}{\mathsf{ki}(D,E,\mathsf{ki}(A,B,C))}\ \mathsf{ki}_{10}^{\mathsf{ki}}$$

$$\frac{\mathsf{ki}(E,F,\mathsf{ki}(A,B,\mathsf{ki}(A,C,D)))}{\mathsf{ki}(E,F,\mathsf{ki}(A,C,\mathsf{ki}(A,B,D)))}\ \mathsf{ki}_{12}^{\mathsf{ki}}$$

$$\frac{\mathsf{ki}(G,H,\mathsf{ki}(B,F,A))}{\mathsf{ki}(G,H,\mathsf{ki}(B,F,\mathsf{ki}(A,\mathsf{ki}(A,C,\mathsf{ki}(A,D,E)),\mathsf{ki}(A,\mathsf{ki}(A,C,D),\mathsf{ki}(A,C,E)))))}\ \mathsf{ki}_{3}^{\mathsf{ki}}$$

$$\frac{\mathsf{ki}(F,G,\mathsf{ki}(B,E,A))}{\mathsf{ki}(F,G,\mathsf{ki}(B,E,\mathsf{ki}(A,\mathsf{ki}(A,\mathsf{ki}(A,C,D),C),C)))}\ \mathsf{ki}_{4}^{\mathsf{ki}}$$

$$\frac{\mathsf{ki}(E,F,\mathsf{ki}(A,B,\mathsf{ki}(A,C,D)))}{\mathsf{ki}(E,F,\mathsf{ki}(A,\mathsf{ki}(B,B,C),D))}\ \mathsf{ki}_{5}^{\mathsf{ki}}$$

$$\frac{\mathsf{ki}(E,F,\mathsf{ki}(A,\mathsf{ki}(B,B,C),D))}{\mathsf{ki}(E,F,\mathsf{ki}(A,B,\mathsf{ki}(A,C,D)))}\ \mathsf{ki}_{6}^{\mathsf{ki}}$$

$$\frac{\mathsf{ki}(F,G,\mathsf{ki}(A,E,B)) \quad \mathsf{ki}(F,G,\mathsf{ki}(A,E,\mathsf{ki}(A,C,D)))}{\mathsf{ki}(F,G,\mathsf{ki}(A,E,\mathsf{ki}(B,C,D)))}\ \mathsf{ki}_{7}^{\mathsf{ki}}$$

$$\frac{\mathsf{ki}(F,G,\mathsf{ki}(A,E,\mathsf{ki}(B,C,D)))}{\mathsf{ki}(F,G,\mathsf{ki}(A,E,B))}\ \mathsf{ki}_{8}^{\mathsf{ki}}$$

$$\frac{\mathsf{ki}(F,G,\mathsf{ki}(A,E,\mathsf{ki}(B,C,D)))}{\mathsf{ki}(F,G,\mathsf{ki}(A,E,\mathsf{ki}(A,C,D)))}\ \mathsf{ki}_{9}^{\mathsf{ki}}$$

Proof. The formally verified derivation of each rule is presented below, following what was explained in Chapter 3.

- ki_{10}

```
theorem ki10 {a b c : Prop} (h1 : a) (h2 : c) : ki a b c :=
  have h3 : ki a c (ki a b c), from ki2 h1,
  show ki a b c, from ki1 h2 h3
```

- ki'_{3}

```
theorem ki'3 {a b c d : Prop} (h1 : a) :
  ki a (ki a b (ki a c d)) (ki a (ki a b c) (ki a b d)) :=
  have h2 : ki a a a, from ki10 h1 h1,
  have h3 : ki a a (ki a (ki a b (ki a c d)) (ki a (ki a b c) (ki a b d))), from ki3 h2,
```

```
  show ki a (ki a b (ki a c d)) (ki a (ki a b c) (ki a b d)), from ki₁ h₁ h₃
```

- ki'_4

```
theorem ki'₄ {a b c : Prop} (h₁ : a) : ki a (ki a (ki a c b) c) c :=
  have h₂ : ki a a a, from ki₁₀ h₁ h₁,
  have h₃ : ki a a (ki a (ki a (ki a c b) c) c), from ki₄ h₂,
  show ki a (ki a (ki a c b) c) c, from ki₁ h₁ h₃
```

- ki_0

```
theorem ki₀ {a b c : Prop} (h₁ : ki a b c) : a :=
  let r := ki a b c in
    have h₂ : r, from h₁,
    have h₃ : ki r r (ki a b c), from ki₁₀ h₂ h₂,
    have h₄ : ki r r a, from ki₈ h₃,
    show a, from ki₁ h₂ h₄
```

- ki_{11}

```
theorem ki₁₁ {a b : Prop} (h₁ : a) : ki a b b :=
  have h₂ : ki a (ki a b (ki a a b)) (ki a (ki a b a) (ki a b b)), from ki'₃ h₁,
  have h₃ : ki a b (ki a a b), from ki₂ h₁,
  have h₄ : ki a (ki a b a) (ki a b b), from ki₁ h₃ h₂,
  have h₅ : ki a b a, from ki₁₀ h₁ h₁,
  show ki a b b, from ki₁ h₅ h₄
```

- ki_{12}

```
theorem ki₁₂ {a b c d : Prop} (h₁ : ki a b (ki a c d)) : ki a c (ki a b d) :=
  have h₂ : a, from ki₀ h₁,
  have h₃ : ki a (ki a b (ki a c d)) (ki a (ki a b c) (ki a b d)), from ki'₃ h₂,
  have h₄ : ki a (ki a b c) (ki a b d), from ki₁ h₁ h₃,
  have h₅ : ki a c (ki a (ki a b c) (ki a b d)), from ki₁₀ h₂ h₄,
  let r := ki a b d, q := ki a b c in
    have h₆ : ki a (ki a c (ki a q r)) (ki a (ki a c q) (ki a c r)), from ki'₃ h₂,
    have h₇ : ki a (ki a c q) (ki a c r), from ki₁ h₅ h₆,
    have h₈ : ki a c q, from ki₂ h₂,
    have h₉ : ki a c r, from ki₁ h₈ h₇,
    show ki a c (ki a b d), from h₉
```

- ki_{13}

```
theorem ki₁₃ {a b c : Prop} (h₁ : ki a b (ki a b c)) : ki a b c :=
  have h₂ : a, from ki₀ h₁,
  have h₃ : ki a (ki a b (ki a b c)) (ki a (ki a b b) (ki a b c)), from ki'₃ h₂,
  have h₄ : ki a (ki a b b) (ki a b c), from ki₁ h₁ h₃,
  have h₅ : ki a b b, from ki₁₁ h₂,
  show ki a b c, from ki₁ h₅ h₄
```

- ki_{14}

```
theorem ki₁₄ {a b c d : Prop} (h₁ : ki a b d) : ki a (ki b b c) d :=
  have h₂ : a, from ki₀ h₁,
  have h₃ : ki a c (ki a b d), from ki₁₀ h₂ h₁,
  have h₄ : ki a b (ki a c d), from ki₁₂ h₃,
  show ki a (ki b b c) d, from ki₅ h₄
```

- ki_{15}

```
theorem ki₁₅ {a b c d : Prop} (h₁ : ki a b d) : ki a (ki c c b) d :=
  have h₂ : a, from ki₀ h₁,
  have h₃ : ki a c (ki a b d), from ki₁₀ h₂ h₁,
  show ki a (ki c c b) d, from ki₅ h₃
```

- ki_{16}

```
theorem ki₁₆ {a b c d : Prop} (h₁ : ki a b c) (h₂ : ki a c d) : ki a b d :=
  have h₃ : a, from ki₀ h₁,
  have h₄ : ki a (ki a b (ki a c d)) (ki a (ki a b c) (ki a b d)), from ki'₃ h₃,
  have h₅ : ki a b (ki a c d), from ki₁₀ h₃ h₂,
  have h₆ : ki a (ki a b c) (ki a b d), from ki₁ h₅ h₄,
  show ki a b d, from ki₁ h₁ h₆
```

- ki_1^{ki}

```
theorem ki₁_ki {a b c d e : Prop} (h₁ : ki d e b) (h₂ : ki d e (ki a b c)) : ki d e c :=
  have h₃ : ki d e (ki d b c), from ki₉ h₂,
  have h₄ : ki d b (ki d e c), from ki₁₂ h₃,
```

```
  have h₅ : ki d e (ki d e c), from ki₁₆ h₁ h₄,
  show ki d e c, from ki₁₃ h₅
```

- $\mathsf{ki}_2^{\mathsf{ki}}$

```
theorem ki₂_ki {a b c d e : Prop} (h₁ : ki d e a) : ki d e (ki a b (ki a c b)) :=
  have h₂ : d, from ki₀ h₁,
  have h₃ : ki d b (ki d c b), from ki₂ h₂,
  have h₄ : ki d (ki e e b) (ki d c b), from ki₁₅ h₃,
  have h₅ : ki d (ki e e b) a, from ki₁₄ h₁,
  have h₆ : ki d (ki e e b) (ki a c b), from ki₇ h₅ h₄,
  have h₇ : ki d e (ki d b (ki a c b)), from ki₆ h₆,
  show ki d e (ki a b (ki a c b)), from ki₇ h₁ h₇
```

- $\mathsf{ki}_{10}^{\mathsf{ki}}$

```
theorem ki₁₀_ki {a b c d e : Prop} (h₁ : ki d e a) (h₂ : ki d e c) : ki d e (ki a b c) :=
  have h₃ : ki d e (ki a c (ki a b c)), from ki₂_ki h₁,
  show ki d e (ki a b c), from ki₁_ki h₂ h₃
```

- $\mathsf{ki}_{12}^{\mathsf{ki}}$

```
theorem ki₁₂_ki {a b c d e f : Prop} (h₁ : ki e f (ki a b (ki a c d))) :
    ki e f (ki a c (ki a b d)) :=
  have h₂ : ki e f a, from ki₈ h₁,
  have h₃ : ki e f (ki a (ki a b (ki a c d)) (ki a (ki a b c) (ki a b d))), from ki₃ h₂,
  have h₄ : ki e f (ki a (ki a b c) (ki a b d)), from ki₁_ki h₁ h₃,
  have h₅ : ki e f (ki a c (ki a (ki a b c) (ki a b d))), from ki₁₀_ki h₂ h₄,
  let r := ki a b d, q := ki a b c in
    have h₆ : ki e f (ki a (ki a c (ki a q r)) (ki a (ki a c q) (ki a c r))), from ki₃ h₂,
    have h₇ : ki e f (ki a (ki a c q) (ki a c r)), from ki₁_ki h₅ h₆,
    have h₈ : ki e f (ki a c q), from ki₂_ki h₂,
    have h₉ : ki e f (ki a c r), from ki₁_ki h₈ h₇,
    show ki e f (ki a c (ki a b d)), from h₉
```

- $\mathsf{ki}_3^{\mathsf{ki}}$

```
theorem ki₃_ki {a b c d e f g h : Prop}
  (h₁ : ki g h (ki b f a)) :
  ki g h (ki b f (ki a (ki a c (ki a d e)) (ki a (ki a c d) (ki a c e)))) :=
```

```
  have h₂ : ki g h (ki g f a), from ki₉ h₁,
  have h₃ : ki g (ki h h f) a, from ki₅ h₂,
  have h₄ : ki g (ki h h f) (ki a (ki a c (ki a d e)) (ki a (ki a c d) (ki a c e))), from ki₃ h₃,
  have h₅ : ki g h (ki g f (ki a (ki a c (ki a d e)) (ki a (ki a c d) (ki a c e)))), from ki₆ h₄,
  have h₆ : ki g h b, from ki₈ h₁,
  show ki g h (ki b f (ki a (ki a c (ki a d e)) (ki a (ki a c d) (ki a c e)))), from ki₇ h₆ h₅
```

- ki_4^{ki}

```
theorem ki₄_ki {a b c d e f g : Prop}
  (h₁ : ki f g (ki b e a)) :
  ki f g (ki b e (ki a (ki a (ki a c d) c) c)) :=
  have h₂ : ki f g (ki f e a), from ki₉ h₁,
  have h₃ : ki f (ki g g e) a, from ki₅ h₂,
  have h₄ : ki f (ki g g e) (ki a (ki a (ki a c d) c) c), from ki₄ h₃,
  have h₅ : ki f g (ki f e (ki a (ki a (ki a c d) c) c)), from ki₆ h₄,
  have h₆ : ki f g b, from ki₈ h₁,
  show ki f g (ki b e (ki a (ki a (ki a c d) c) c)), from ki₇ h₆ h₅
```

- ki_5^{ki}

```
theorem ki₅_ki {a b c d e f : Prop} (h₁ : ki e f (ki a b (ki a c d))) :
  ki e f (ki a (ki b b c) d) :=
  have h₂ : ki e f (ki e b (ki a c d)), from ki₉ h₁,
  have h₃ : ki e b (ki e f (ki a c d)), from ki₁₂ h₂,
  have h₄ : ki e (ki b b f) (ki a c d), from ki₅ h₃,
  have h₅ : ki e (ki b b f) (ki e c d), from ki₉ h₄,
  have h₆ : ki e b (ki e f (ki e c d)), from ki₆ h₅,
  have h₇ : ki e b (ki e c (ki e f d)), from ki₁₂_ki h₆,
  have h₈ : ki e (ki b b c) (ki e f d), from ki₅ h₇,
  have h₉ : ki e f (ki e (ki b b c) d), from ki₁₂ h₈,
  have h₁₀ : ki e f a, from ki₈ h₁,
  show ki e f (ki a (ki b b c) d), from ki₇ h₁₀ h₉
```

- ki_6^{ki}

```
theorem ki₆_ki {a b c d e f : Prop} (h₁ : ki e f (ki a (ki b b c) d)) :
  ki e f (ki a b (ki a c d)) :=
  have h₂ : ki e f (ki e (ki b b c) d), from ki₉ h₁,
  have h₃ : ki e (ki b b c) (ki e f d), from ki₁₂ h₂,
  have h₄ : ki e b (ki e c (ki e f d)), from ki₆ h₃,
```

```
  have h₅ : ki e b (ki e f (ki e c d)), from ki₁₂_ki h₄,
  have h₆ : ki e f (ki e b (ki e c d)), from ki₁₂ h₅,
  have h₇ : ki e (ki f f b) (ki e c d), from ki₅ h₆,
  have h₈ : ki e f a, from ki₈ h₁,
  have h₉ : ki e (ki f f b) a, from ki₁₄ h₈,
  have h₁₀ : ki e (ki f f b) (ki a c d), from ki₇ h₉ h₇,
  have h₁₁ : ki e f (ki e b (ki a c d)), from ki₆ h₁₀,
  show ki e f (ki a b (ki a c d)), from ki₇ h₈ h₁₁
```

- $\mathsf{ki}_7^{\mathsf{ki}}$

```
theorem ki₇_ki {a b c d e f g : Prop}
  (h₁ : ki f g (ki a e b))
  (h₂ : ki f g (ki a e (ki a c d))) :
  ki f g (ki a e (ki b c d)) :=
  have h₃ : ki f g (ki f e b), from ki₉ h₁,
  have h₄ : ki f (ki g g e) b, from ki₅ h₃,
  have h₅ : ki f g a, from ki₈ h₁,
  have h₆ : ki f g (ki f e (ki a c d)), from ki₉ h₂,
  have h₇ : ki f (ki g g e) (ki a c d), from ki₅ h₆,
  have h₈ : ki f (ki g g e) (ki f c d), from ki₉ h₇,
  have h₉ : ki f (ki g g e) (ki b c d), from ki₇ h₄ h₈,
  have h₁₀ : ki f g (ki f e (ki b c d)), from ki₆ h₉,
  show ki f g (ki a e (ki b c d)), from ki₇ h₅ h₁₀
```

- $\mathsf{ki}_8^{\mathsf{ki}}$

```
theorem ki₈_ki {a b c d e f g : Prop} (h₁ : ki f g (ki a e (ki b c d))) :
  ki f g (ki a e b) :=
  have h₂ : ki f g (ki f e (ki b c d)), from ki₉ h₁,
  have h₃ : ki f (ki g g e) (ki b c d), from ki₅ h₂,
  have h₄ : ki f (ki g g e) b, from ki₈ h₃,
  have h₅ : ki f g (ki f e b), from ki₆ h₄,
  have h₆ : ki f g a, from ki₈ h₁,
  show ki f g (ki a e b), from ki₇ h₆ h₅
```

- $\mathsf{ki}_9^{\mathsf{ki}}$

```
theorem ki₉_ki {a b c d e f g : Prop} (h₁ : ki f g (ki a e (ki b c d))) :
  ki f g (ki a e (ki a c d)) :=
  have h₂ : ki f g a, from ki₈ h₁,
```

```
have h₃ : ki f g (ki f e (ki b c d)), from ki₉ h₁,
have h₄ : ki f (ki g g e) (ki b c d), from ki₅ h₃,
have h₅ : ki f (ki g g e) (ki f c d), from ki₉ h₄,
have h₆ : ki f (ki g g e) a, from ki₁₄ h₂,
have h₇ : ki f (ki g g e) (ki a c d), from ki₇ h₆ h₅,
have h₈ : ki f g (ki f e (ki a c d)), from ki₆ h₇,
show ki f g (ki a e (ki a c d)), from ki₇ h₂ h₈
```

□

Having the ki-lifted versions of each primitive rule of $\mathscr{B}_{\mathsf{ki}}$, we are ready to prove a deduction theorem for this calculus.

Lemma 4.4.3. *The following property holds for* $\vdash_{\mathscr{B}_{\mathsf{ki}}}$*:*

$$\text{for all } \Gamma \cup \{\mathrm{A}, \mathrm{B}, \mathrm{C}\} \subseteq L_{\mathsf{ki}}.\ \text{if } \Gamma, \mathrm{A}, \mathrm{B} \vdash_{\mathscr{B}_{\mathsf{ki}}} \mathrm{C} \text{ then } \Gamma, \mathrm{A} \vdash_{\mathscr{B}_{\mathsf{ki}}} \mathsf{ki}(\mathrm{A}, \mathrm{B}, \mathrm{C}) \qquad (\delta_{\mathsf{ki}})$$

Proof. Let $\Gamma \cup \{\mathrm{A}, \mathrm{B}, \mathrm{C}\} \subseteq L_{\mathsf{ki}}$. The proof goes by induction on the derivation of C from $\Gamma \cup \{\mathrm{A}, \mathrm{B}\}$. In this way, suppose that $\Gamma, \mathrm{A}, \mathrm{B} \vdash_{\mathscr{B}_{\mathsf{ki}}} \mathrm{C}$, and that this is witnessed by a formal proof consisting of the sequence of formulas $\mathrm{D}_1, \ldots, \mathrm{D}_n$, where $n \geq 1$, with $\mathrm{D}_n = \mathrm{C}$. We will prove that $\Gamma, \mathrm{A} \vdash_{\mathscr{B}_{\mathsf{ki}}} \mathsf{ki}(\mathrm{A}, \mathrm{B}, \mathrm{D}_j)$ for all $1 \leq j \leq n$. For the base case, when $n = 1$, there are three cases. First, when $\mathrm{D}_1 \in \Gamma$, we have $\Gamma, \mathrm{A} \vdash_{\mathscr{B}_{\mathsf{ki}}} \mathsf{ki}(\mathrm{A}, \mathrm{B}, \mathrm{D}_1)$ by ki_{10} from the facts that $\Gamma, \mathrm{A} \vdash_{\mathscr{B}_{\mathsf{ki}}} \mathrm{A}$ (by (R) and (M)) and $\Gamma, \mathrm{A} \vdash_{\mathscr{B}_{\mathsf{ki}}} \mathrm{D}_1$ (since $\mathrm{D}_1 \in \Gamma$ by hypothesis). Second, when D_1 is A, the previous reasoning works similarly. Third, when D_1 is B, by ki_{11} applied to A, we get $\mathsf{ki}(\mathrm{A}, \mathrm{B}, \mathrm{B})$, which translates to $\mathsf{ki}(\mathrm{A}, \mathrm{B}, \mathrm{D}_1)$, since B is D_1 by hypothesis. Now, in the induction step, suppose that $\Gamma, \mathrm{A} \vdash_{\mathscr{B}_{\mathsf{ki}}} \mathsf{ki}(\mathrm{A}, \mathrm{B}, \mathrm{D}_k)$ for all $k < j$ and some $j > 1$. If $\mathrm{D}_j \in \Gamma \cup \{\mathrm{A}, \mathrm{B}\}$, then the proof is analogous to the one for the base case. Otherwise, D_j results from the application of some instance $\langle \mathrm{D}_{k_1}, \ldots, \mathrm{D}_{k_m}, \mathrm{D}_j \rangle$, where $k_l < j$ for $1 \leq l < j$, of one m-ary primitive rule of $\mathscr{B}_{\mathsf{ki}}$, say ki_l. From the induction hypothesis and the corresponding ki-lifted version, namely $\mathsf{ki}_l^{\mathsf{ki}}$, proved to be derivable in Lemma 4.4.2, we have that $\mathsf{ki}(\mathrm{A}, \mathrm{B}, \mathrm{D}_j)$ is provable from $\Gamma \cup \{\mathrm{A}\}$. The desired result is the case $j = n$. □

Lemma 4.4.4. *Every set* Γ^+ *that is* Z*-maximal with respect to* $\vdash_{\mathscr{B}_{\mathsf{ki}}}$ *is maximal (consistent).*

Proof. Suppose that $\Gamma^+ \subseteq L_{\mathsf{ki}}$ is Z-maximal and assume that $\mathrm{A} \notin \Gamma^+$. Our goal is to prove that $\Gamma^+, \mathrm{A} \vdash_{\mathscr{B}_{\mathsf{ki}}} \mathrm{B}$ for any $\mathrm{B} \in L_{\mathsf{ki}}$. First of all, take $\mathrm{C} \in \Gamma^+$ (Lemma 2.6.2 allows us to do this). A fundamental step for proving this lemma is showing that $\Gamma^+ \vdash_{\mathscr{B}_{\mathsf{ki}}} \mathsf{ki}(C, Z, B)$. In this direction, suppose, for the sake of contradiction, that $\Gamma^+ \nvdash_{\mathscr{B}_{\mathsf{ki}}} \mathsf{ki}(C, Z, B)$. The following reasoning proves the desired result by deriving an absurd:

(1)	$\Gamma^+, \mathsf{ki}(C, Z, B) \vdash_{\mathscr{B}_{\mathsf{ki}}} Z$	Lemma 2.6.1.1
(2)	$\Gamma^+, C, \mathsf{ki}(C, Z, B) \vdash_{\mathscr{B}_{\mathsf{ki}}} Z$	1 (M)
(3)	$\Gamma^+, C \vdash_{\mathscr{B}_{\mathsf{ki}}} \mathsf{ki}(C, \mathsf{ki}(C, Z, B), Z)$	2 δ_{ki}
(4)	$C \vdash_{\mathscr{B}_{\mathsf{ki}}} C$	(R)
(5)	$\Gamma^+, C \vdash_{\mathscr{B}_{\mathsf{ki}}} C$	4 (M)
(6)	$\Gamma^+, C \vdash_{\mathscr{B}_{\mathsf{ki}}} \mathsf{ki}(C, \mathsf{ki}(C, \mathsf{ki}(C, Z, B), Z), Z)$	5 ki'_4
(7)	$\Gamma^+, C \vdash_{\mathscr{B}_{\mathsf{ki}}} Z$	3, 6 ki_1
(8)	$\Gamma^+ \vdash_{\mathscr{B}_{\mathsf{ki}}} Z$	7, $C \in \Gamma^+$

Because $\Gamma^+ \vdash_{\mathscr{B}_{\mathsf{ki}}} \mathsf{ki}(C, Z, B)$, we get (a): $\Gamma^+, A \vdash_{\mathscr{B}_{\mathsf{ki}}} \mathsf{ki}(C, Z, B)$ by (M). Since $A \notin \Gamma^+$, we also have (b): $\Gamma^+, A \vdash_{\mathscr{B}_{\mathsf{ki}}} Z$. Then these consecutions, by ki_1, yield $\Gamma^+, A \vdash_{\mathscr{B}_{\mathsf{ki}}} B$, proving that Γ^+ is maximal. □

Theorem 4.4.5. *The calculus $\mathscr{B}_{\mathsf{ki}}$ is complete with respect to the matrix $\mathfrak{Q}_{\mathsf{ki}}$.*

Proof. Let $\Gamma \cup \{Z\} \subseteq L_{\mathsf{ki}}$ such that $\Gamma \nvdash_{\mathscr{B}_{\mathsf{ki}}} Z$ and take a Z-maximal theory $\Gamma^+ \supseteq \Gamma$ by the Lindenbaum-Asser Lemma. From the truth-table of ki and the formulation given in Section 2.7, the completeness property (ki) is given by:

$$\mathsf{ki}(A, B, C) \in \Gamma^+ \text{ iff } A \in \Gamma^+ \text{ and } (B \notin \Gamma^+ \text{ or } C \in \Gamma^+) \qquad (\mathsf{ki})$$

In the left-to-right direction, suppose that (a): $\mathsf{ki}(A, B, C) \in \Gamma^+$. By ki_0, $\Gamma^+ \vdash_{\mathscr{B}_{\mathsf{ki}}} A$, so $A \in \Gamma^+$. By cases, the desired result is directly obtained when $B \notin \Gamma^+$, and, if $B \in \Gamma^+$, then (b): $\Gamma^+ \vdash_{\mathscr{B}_{\mathsf{ki}}} B$, and, by ki_1 from (a) and (b), $\Gamma^+ \vdash_{\mathscr{B}_{\mathsf{ki}}} C$, hence $C \in \Gamma^+$ as desired. From the right to the left, suppose that $A \in \Gamma^+$, thus (c): $\Gamma^+ \vdash_{\mathscr{B}_{\mathsf{ki}}} A$. Let us work again by cases. Suppose that $B \notin \Gamma^+$. Then, Lemma 4.4.4 implies that $\Gamma^+, B \vdash_{\mathscr{B}_{\mathsf{ki}}} C$, and, by (M), $\Gamma^+, A, B \vdash_{\mathscr{B}_{\mathsf{ki}}} C$. By δ_{ki}, $\Gamma^+, A \vdash_{\mathscr{B}_{\mathsf{ki}}} \mathsf{ki}(A, B, C)$, and, because of (c), we conclude that $\Gamma^+ \vdash_{\mathscr{B}_{\mathsf{ki}}} \mathsf{ki}(A, B, C)$. Now, if $C \in \Gamma^+$, then (d): $\Gamma^+ \vdash_{\mathscr{B}_{\mathsf{ki}}} C$. By ki_{10} from (c) and (d), we get $\Gamma^+ \vdash_{\mathscr{B}_{\mathsf{ki}}} \mathsf{ki}(A, B, C)$, so $\mathsf{ki}(A, B, C) \in \Gamma^+$. □

Remark 4.4.1. Similarly to what occurs for the calculus $\mathscr{B}_{\mathsf{ka}}$, if a new rule r is added to $\mathscr{B}_{\mathsf{ki}}$, then deriving its ki-lifted version, namely r^{ki}, causes the completeness property (ki) to be preserved in the expanded calculus.

The proposed calculus for the expansion $\mathcal{B}_{\mathsf{ki},\bot}$ results from adding an interaction rule to $\mathscr{B}_{\mathsf{ki}}$, as presented below:

Hilbert Calculus 12. $\mathscr{B}_{\mathsf{ki},\bot}$

$$\mathscr{B}_{\mathsf{ki}} \quad \frac{\mathsf{ki}(\mathrm{A}, \mathrm{B}, \bot)}{\mathsf{ki}(\mathrm{A}, \mathrm{B}, \mathrm{C})}\ \mathsf{kib}_1$$

Theorem 4.4.6. *The calculus $\mathscr{B}_{\mathsf{ki},\bot}$ is sound with respect to the matrix $\mathfrak{2}_{\mathsf{ki},\bot}$.*

Proof. Since the rules of $\mathscr{B}_{\mathsf{ki}}$ are sound with respect to $\mathfrak{2}_{\mathsf{ki}}$ (see Theorem 4.4.1), it remains to prove soundness for rule kib_1 with respect to $\mathfrak{2}_{\mathsf{ki},\bot}$. Consider v an arbitrary $\mathfrak{2}_{\mathsf{ki},\bot}$-valuation such that $v(\mathsf{ki}(\mathrm{A}, \mathrm{B}, \bot)) = 1$, then $v(\mathrm{A}) = 1$ and $v(\mathrm{B}) = 0$, forcing the conclusion to be evaluated to 1. □

The completeness proof in this case is analogous to that of $\mathscr{B}_{\mathsf{ka},\bot}$: we need to derive the ki-lifted version of the new rule, as well as rule b_1.

Lemma 4.4.7. *The following rules are derivable in $\mathscr{B}_{\mathsf{ki},\bot}$:*

$$\frac{\mathsf{ki}(\mathrm{D}, \mathrm{E}, \mathsf{ki}(\mathrm{A}, \mathrm{B}, \bot))}{\mathsf{ki}(\mathrm{D}, \mathrm{E}, \mathsf{ki}(\mathrm{A}, \mathrm{B}, \mathrm{C}))}\ \mathsf{kib}_1^{\mathsf{ki}}$$

$$\frac{\bot}{\mathrm{A}}\ \mathsf{b}_1$$

Proof. The formally verified derivation of each rule is presented below:

- $\mathsf{kib}_1^{\mathsf{ki}}$

```
theorem kib₁_ki {a b c d e : Prop} (h₁ : ki d e (ki b a bot)) : ki d e (ki b a c) :=
  have h₂ : ki d e (ki d a bot), from ki.ki₉ h₁,
  have h₃ : ki d (ki e e a) bot, from ki.ki₅ h₂,
  have h₄ : ki d (ki e e a) c, from kib₁ h₃,
  have h₅ : ki d e (ki d a c), from ki.ki₆ h₄,
  have h₆ : ki d e b, from ki.ki₈ h₁,
  show ki d e (ki b a c), from ki.ki₇ h₆ h₅
```

- b_1

```
theorem b₁ {a : Prop} (h₁ : bot) : a :=
  have h₂ : ki bot bot bot, from ki.ki₁₀ h₁ h₁,
  have h₃ : ki bot bot a, from kib₁ h₂,
  show a, from ki.ki₁ h₁ h₃
```

□

Theorem 4.4.8. *The calculus* $\mathscr{B}_{\mathsf{ki},\bot}$ *is complete with respect to the matrix* $2_{\mathsf{ki},\bot}$.

Proof. The derived rules b_1 and $\mathsf{kib}_1^{\mathsf{ki}}$, in view of Remark 4.1.1 and Remark 4.3.1, imply the completeness properties ($\bot$) and (ki), respectively. □

4.4.2 On the expansions of $\mathcal{B}_{\mathsf{ki}}$ and $\mathcal{B}_{\mathsf{ki},\bot}$

In [11, Section 3], we find results that allow to generate calculi for expansions of some fragments based on known axiomatizations. Although their ultimost consequence is the axiomatizability of the infinite portion of Post's lattice, we can use them to axiomatize some fragments of the finite portion. In this section, we present, without proving, a theorem that allows us to axiomatize logics that expand $\mathcal{B}_{\mathsf{ki}}$, located in the highest part of Post's lattice, covered in Section 4.15. The proof given by Rautenberg provides a clear procedure to construct the calculi for such expansions.

Theorem 4.4.9. *An axiomatization of any expansion of* $\mathcal{B}_{\mathsf{ki}}$ *and* $\mathcal{B}_{\mathsf{ki},\bot}$ *is obtained, respectively, from* $\mathscr{B}_{\mathsf{ki}}$ *and* $\mathscr{B}_{\mathsf{ki},\bot}$ *by adding several at most unary rules.*

Proof. See Theorem 1.1 in [11, p. 336]. □

4.5 $\mathcal{B}_\vee$, $\mathcal{B}_{\vee,\top}$, $\mathcal{B}_{\vee,\bot}$, $\mathcal{B}_{\vee,\bot,\top}$

In this section, we propose axiomatizations for $\mathcal{B}_\vee$ and its expansions by the constants $\top$ and $\bot$. Moreover, some properties about the connective $\vee$ and its rules, to be used in future sections, will be proved. Also, we will give a result that provide a recipe for constructing axiomatizations for any monotonic expansion of $2_\vee$.

Hilbert Calculus 13. $\mathscr{B}_\vee$

$$\frac{\mathrm{A}}{\mathrm{A} \vee \mathrm{B}}\ \mathsf{d}_1 \qquad \frac{\mathrm{A} \vee \mathrm{A}}{\mathrm{A}}\ \mathsf{d}_2 \qquad \frac{\mathrm{A} \vee \mathrm{B}}{\mathrm{B} \vee \mathrm{A}}\ \mathsf{d}_3 \qquad \frac{\mathrm{A} \vee (\mathrm{B} \vee \mathrm{C})}{(\mathrm{A} \vee \mathrm{B}) \vee \mathrm{C}}\ \mathsf{d}_4$$

Theorem 4.5.1. *The calculus $\mathscr{B}_\vee$ is sound with respect to the matrix $\mathfrak{2}_\vee$.*

Proof. Consider the truth-table presenting all possible truth-values of the formulas involved in the rules of $\mathscr{B}_\vee$ under $\mathfrak{2}_\vee$-valuations:

A	B	C	A ∨ B	A ∨ A	B ∨ A	A ∨ (B ∨ C)	(A ∨ B) ∨ C
1	1	1	1	1	1	1	1
1	1	0	1	1	1	1	1
1	0	1	1	1	1	1	1
1	0	0	1	1	1	1	1
0	1	1	1	0	1	1	1
0	1	0	1	0	1	1	1
0	0	1	0	0	0	1	1
0	0	0	0	0	0	0	0

Notice from the table above that it is never the case that the premisses of the rules evaluates to 1 and the conclusion evaluates to 0. □

In what follows, if d is an n-ary rule, with $n \in \omega$, let $\mathsf{d}^\vee$, the $\vee$-lifted version of d, be the rule given by the set of instances $\langle \mathrm{C} \vee \mathrm{A}_1, \ldots, \mathrm{C} \vee \mathrm{A}_n, \mathrm{C} \vee \mathrm{B} \rangle$, where $\langle \mathrm{A}_1, \ldots, \mathrm{A}_n, \mathrm{B} \rangle$ is an instance of d and $\mathrm{C} \in L_\vee$. The next lemma establishes in particular that the $\vee$-lifted versions of the primitive rules of $\mathscr{B}_\vee$ are derivable in this system, a fundamental result for proving the subsequent monotonicity property $m_\vee$, and thus, as we will see, the completeness of this calculus with respect to $\mathfrak{2}_\vee$.

Lemma 4.5.2. *The following rules are derivable in $\mathscr{B}_\vee$:*

$$\frac{\mathrm{B}}{\mathrm{A} \vee \mathrm{B}}\ \mathsf{d}'_1$$

$$\frac{(\mathrm{A} \vee \mathrm{B}) \vee \mathrm{C}}{\mathrm{A} \vee (\mathrm{B} \vee \mathrm{C})}\ \mathsf{d}'_4$$

$$\frac{\mathrm{C} \vee \mathrm{A}}{\mathrm{C} \vee (\mathrm{A} \vee \mathrm{B})}\ \mathsf{d}^\vee_1$$

$$\frac{\mathrm{B} \vee (\mathrm{A} \vee \mathrm{A})}{\mathrm{B} \vee \mathrm{A}}\ \mathsf{d}^\vee_2$$

$$\frac{C \vee (A \vee B)}{C \vee (B \vee A)}\ \mathsf{d}_3^{\vee}$$

$$\frac{D \vee (A \vee (B \vee C))}{D \vee ((A \vee B) \vee C)}\ \mathsf{d}_4^{\vee}$$

Proof. The formally verified derivation of each rule is presented below, following what was explained in Chapter 3.

- d_1'

```
theorem d₁' {a b : Prop} (h₁ : b) : or a b :=
  have h₂ : or b a, from or.d₁ h₁,
  show or a b, from or.d₃ h₂
```

- d_4'

```
theorem d₄' {a b c : Prop} (h₁ : or (or a b) c) : or a (or b c) :=
  have h₂ : or c (or a b), from or.d₃ h₁,
  have h₃ : or (or c a) b, from or.d₄ h₂,
  have h₄ : or b (or c a), from or.d₃ h₃,
  have h₅ : or (or b c) a, from or.d₄ h₄,
  show or a (or b c), from or.d₃ h₅
```

- $\mathsf{d}_1^{\vee}$

```
theorem d₁_or {a b c : Prop} (h₁ : or c a) : or c (or a b) :=
  have h₂ : or (or c a) b, from or.d₁ h₁,
  show or c (or a b), from or.d₄' h₂
```

- $\mathsf{d}_2^{\vee}$

```
theorem d₂_or {a b : Prop} (h₁ : or b (or a a)) : or b a :=
  have h₂ : or (or b a) a, from or.d₄ h₁,
  have h₃ : or a (or b a), from or.d₃ h₂,
  have h₄ : or b (or a (or b a)), from or.d₁' h₃,
  have h₅ : or (or b a) (or b a), from or.d₄ h₄,
  show or b a, from or.d₂ h₅
```

- $\mathsf{d}_3^{\vee}$

```
theorem d₃_or {a b c : Prop} (h₁ : or c (or a b)) : or c (or b a) :=
  have h₂ : or (or c a) b, from or.d₄ h₁,
  have h₃ : or (or (or c a) b) a, from or.d₁ h₂,
  have h₄ : or (or c a) (or b a), from or.d₄' h₃,
  have h₅ : or c (or a (or b a)), from or.d₄' h₄,
  have h₆ : or (or a (or b a)) c, from or.d₃ h₅,
  have h₇ : or a (or (or b a) c), from or.d₄' h₆,
  have h₈ : or b (or a (or (or b a) c)), from or.d₁' h₇,
  have h₉ : or (or b a) (or (or b a) c), from or.d₄ h₈,
  have h₁₀ : or (or (or b a) (or b a)) c, from or.d₄ h₉,
  have h₁₁ : or c (or (or b a) (or b a)), from or.d₃ h₁₀,
  show or c (or b a), from or.d₂_or h₁₁
```

- $d_4^\vee$

```
theorem d₄_or {a b c d : Prop} (h₁ : or d (or a (or b c))) : or d (or (or a b) c) :=
  have h₂ : or (or d a) (or b c), from or.d₄ h₁,
  have h₃ : or (or d a) (or c b), from or.d₃_or h₂,
  have h₄ : or (or (or d a) c) b, from or.d₄ h₃,
  have h₅ : or (or (or (or d a) c) b) a, from or.d₁ h₄,
  have h₆ : or (or (or d a) c) (or b a), from or.d₄' h₅,
  have h₇ : or (or (or d a) c) (or a b), from or.d₃_or h₆,
  have h₈ : or (or d a) (or c (or a b)), from or.d₄' h₇,
  have h₉ : or (or d a) (or (or a b) c), from or.d₃_or h₈,
  let e := or (or a b) c in
    have h₁₀ : or (or d a) e, from h₉,
    have h₁₁ : or d (or a e), from or.d₄' h₁₀,
    have h₁₂ : or d (or e a), from or.d₃_or h₁₁,
    have h₁₃ : or (or d e) a, from or.d₄ h₁₂,
    have h₁₄ : or (or (or d e) a) b, from or.d₁ h₁₃,
    have h₁₅ : or (or d e) (or a b), from or.d₄' h₁₄,
    have h₁₆ : or (or (or d e) (or a b)) c, from or.d₁ h₁₅,
    have h₁₇ : or (or d e) (or (or a b) c), from or.d₄' h₁₆,
    have h₁₈ : or (or d e) e, from h₁₇,
    have h₁₉ : or d (or e e), from or.d₄' h₁₈,
    have h₂₀ : or d e, from or.d₂_or h₁₉,
    show or d (or (or a b) c), from h₂₀
```

□

Lemma 4.5.3. *The following property holds for* $\vdash_{\mathscr{B}_\vee}$*:*

$$\text{for all } \Gamma \cup \{\mathrm{A}, \mathrm{B}, \mathrm{C}\} \subseteq L_\vee. \text{ if } \Gamma, \mathrm{A} \vdash_{\mathscr{B}_\vee} \mathrm{B} \text{ then } \Gamma, \mathrm{C} \vee \mathrm{A} \vdash_{\mathscr{B}_\vee} \mathrm{C} \vee \mathrm{B} \qquad (m_\vee)$$

Proof. Let $\Gamma \cup \{\mathrm{A}, \mathrm{B}, \mathrm{C}\} \subseteq L_\vee$. Suppose that $\Gamma, \mathrm{A} \vdash_{\mathscr{B}_\vee} \mathrm{B}$ and that this is established by a proof consisting of the sequence $\mathrm{P}_1, \ldots, \mathrm{P}_n = \mathrm{B}$ of formulas, for some $n \in \omega$. The fact that $\Gamma, \mathrm{C} \vee \mathrm{A} \vdash_{\mathscr{B}_\vee} \mathrm{C} \vee \mathrm{P}_j$, for all $1 \leq j \leq n$, can be shown by induction on j. In the base case, $j = 1$, P_1 is either A itself or is a member of Γ. In the first case, since $\mathrm{P}_1 = \mathrm{A}$, $\Gamma, \mathrm{C} \vee \mathrm{A} \vdash_{\mathscr{B}_\vee} \mathrm{C} \vee \mathrm{P}_1$ by an appeal to (R) and (M). In the second case, if $\mathrm{P}_1 \in \Gamma$, by taking assumptions in $\Gamma \cup \{\mathrm{C} \vee \mathrm{A}\}$, the desired conclusion follows simply by the proof below:

(1)	P_1	$\mathrm{P}_1 \in \Gamma$
(2)	$\mathrm{C} \vee \mathrm{P}_1$	$1\ \mathsf{d}_1$

For the inductive step, suppose that $\Gamma, \mathrm{C} \vee \mathrm{A} \vdash_{\mathscr{B}_\vee} \mathrm{C} \vee \mathrm{P}_k$ holds for all $k < j$. Then, either (a): P_j is A, or (b): it is in Γ or (c): it follows by the application of an instance $\langle \mathrm{P}_{k_1}, \ldots, \mathrm{P}_{k_m}, \mathrm{P}_j \rangle$ of some m-ary primitive rule, say d, to premisses $\mathrm{P}_{k_1}, \ldots, \mathrm{P}_{k_m}$, $k_l < j$, for all $1 \leq l \leq m$. Cases (a) and (b) follow by the same arguments used in the proof of the base case. For case (c), notice that the formulas $\mathrm{C} \vee \mathrm{P}_{k_1}, \ldots, \mathrm{C} \vee \mathrm{P}_{k_m}$ follow from $\Gamma \cup \{\mathrm{C} \vee \mathrm{A}\}$ by the inductive hypothesis. Using those formulas, by the corresponding derived rule $\mathsf{d}^\vee$ presented in Lemma 4.5.2, one gets $\mathrm{C} \vee \mathrm{P}_j$. The case where $j = n$ is the desired one for the present proof. □

The above result can be extended to a form that will be useful in Section 4.7. Before going to it, let $\mathrm{B} \vee^0 \mathrm{A} := \mathrm{A}$ and $\mathrm{B} \vee^{n+1} \mathrm{A} := \mathrm{B} \vee (\mathrm{B} \vee^n \mathrm{A})$, where $\mathrm{A}, \mathrm{B} \in L_\vee$ and $n \in \omega$, and consider the following lemma:

Lemma 4.5.4. *For any* $\mathrm{A}, \mathrm{B} \in L_\vee$ *and* $n \in \omega$, $\mathrm{B} \vee^n \mathrm{A} \vdash_{\mathscr{B}_\vee} \mathrm{B} \vee \mathrm{A}$.

Proof. Proceed by induction on n. The base case ($n = 0$) follows by rule d'_1. In the inductive step, suppose that (IH): $\mathrm{B} \vee^k \mathrm{A} \vdash_{\mathscr{B}_\vee} \mathrm{B} \vee \mathrm{A}$, for some $k > 0$. Since $\mathrm{B} \vee^{k+1} \mathrm{A} =$

$\mathrm{B} \vee (\mathrm{B} \vee^k \mathrm{A})$, the following reasoning establishes the desired result:

(1)	$\mathrm{B} \vee (\mathrm{B} \vee^k \mathrm{A}) \vdash_{\mathscr{B}_\vee} \mathrm{B} \vee (\mathrm{B} \vee^k \mathrm{A})$	(R)
(2)	$\mathrm{B} \vee (\mathrm{B} \vee^k \mathrm{A}) \vdash_{\mathscr{B}_\vee} \mathrm{B} \vee (\mathrm{B} \vee (\mathrm{B} \vee^{k-1} \mathrm{A}))$	1 Definition of $\vee^n$
(3)	$\mathrm{B} \vee (\mathrm{B} \vee^k \mathrm{A}) \vdash_{\mathscr{B}_\vee} (\mathrm{B} \vee \mathrm{B}) \vee (\mathrm{B} \vee^{k-1} \mathrm{A})$	2 d_4
(4)	$\mathrm{B} \vee (\mathrm{B} \vee^k \mathrm{A}) \vdash_{\mathscr{B}_\vee} (\mathrm{B} \vee^{k-1} \mathrm{A}) \vee (\mathrm{B} \vee \mathrm{B})$	3 d_3
(5)	$\mathrm{B} \vee (\mathrm{B} \vee^k \mathrm{A}) \vdash_{\mathscr{B}_\vee} (\mathrm{B} \vee^{k-1} \mathrm{A}) \vee \mathrm{B}$	4 $\mathsf{d}_2^\vee$
(6)	$\mathrm{B} \vee (\mathrm{B} \vee^k \mathrm{A}) \vdash_{\mathscr{B}_\vee} \mathrm{B} \vee (\mathrm{B} \vee^{k-1} \mathrm{A})$	5 d_3
(7)	$\mathrm{B} \vee (\mathrm{B} \vee^k \mathrm{A}) \vdash_{\mathscr{B}_\vee} \mathrm{B} \vee^k \mathrm{A}$	6 Definition of $\vee^n$
(8)	$\mathrm{B} \vee^k \mathrm{A} \vdash_{\mathscr{B}_\vee} \mathrm{B} \vee \mathrm{A}$	(IH)
(9)	$\mathrm{B} \vee (\mathrm{B} \vee^k \mathrm{A}) \vdash_{\mathscr{B}_\vee} \mathrm{B} \vee \mathrm{A}$	7, 8 (T)

□

Corollary 4.5.4.1. *If* $\Gamma, \mathrm{A} \vdash_{\mathscr{B}_\vee} \mathrm{B}$*, then* $\Gamma^\vee, \mathrm{C} \vee \mathrm{A} \vdash_{\mathscr{B}_\vee} \mathrm{C} \vee \mathrm{B}$*, where* $\Gamma \cup \{\mathrm{A}, \mathrm{B}, \mathrm{C}\} \subseteq L_\vee$ *and* $\Gamma^\vee := \{\mathrm{C} \vee \mathrm{D} \in L_\vee \mid \mathrm{D} \in \Gamma\}$.

Proof. Let $\Gamma \cup \{\mathrm{A}, \mathrm{B}, \mathrm{C}\} \subseteq L_\vee$ and suppose that $\Gamma, \mathrm{A} \vdash_{\mathscr{B}_\vee} \mathrm{B}$. Since $\vdash_{\mathscr{B}_\vee}$ is finitary, $\Gamma_0, \mathrm{A} \vdash_{\mathscr{B}_\vee} \mathrm{B}$, for some finite $\Gamma_0 \subseteq \Gamma$. Let $n \in \omega$ be the cardinality of Γ_0. Then subsequent applications of $m_\vee$ lead to $\Gamma_0^\vee, \mathrm{C} \vee \mathrm{A} \vdash_{\mathscr{B}_\vee} \mathrm{C} \vee^{n+1} \mathrm{B}$. By Lemma 4.5.4 and (T), we get $\Gamma_0^\vee, \mathrm{C} \vee \mathrm{A} \vdash_{\mathscr{B}_\vee} \mathrm{C} \vee \mathrm{B}$. Finally, because $\Gamma_0^\vee \subseteq \Gamma^\vee$, we get the desired conclusion by (M) applied to the latter consecution. □

We now proceed to the completeness proof of the calculus $\mathscr{B}_\vee$ with respect to $\mathbb{2}_\vee$. For that we need the deduction theorem for disjunction, presented in the next lemma, from which the completeness result follows in a straightforward way.

Lemma 4.5.5. *The following property holds for* $\vdash_{\mathscr{B}_\vee}$*:*

for all $\Gamma \cup \{\mathrm{A}, \mathrm{B}, \mathrm{C}\} \subseteq L_\vee$. if $\Gamma, \mathrm{A} \vdash_{\mathscr{B}_\vee} \mathrm{C}$ and $\Gamma, \mathrm{B} \vdash_{\mathscr{B}_\vee} \mathrm{C}$ then $\Gamma, \mathrm{A} \vee \mathrm{B} \vdash_{\mathscr{B}_\vee} \mathrm{C}$ $\quad (\delta_\vee)$

Proof. Suppose that (h_1): $\Gamma, \mathrm{A} \vdash_{\mathscr{B}_\vee} \mathrm{C}$ and (h_2): $\Gamma, \mathrm{B} \vdash_{\mathscr{B}_\vee} \mathrm{C}$. By ($m_\vee$) applied to ($\mathrm{h}_1$), we obtain the consecution $\Gamma, \mathrm{C} \vee \mathrm{A} \vdash_{\mathscr{B}_\vee} \mathrm{C} \vee \mathrm{C}$, which, by an application of rule d_2, gives (h'$_1$): $\Gamma, \mathrm{C} \vee \mathrm{A} \vdash_{\mathscr{B}_\vee} \mathrm{C}$. Moreover, by $m_\vee$ applied to (h_2), we get the consecution $\Gamma, \mathrm{A} \vee \mathrm{B} \vdash_{\mathscr{B}_\vee} \mathrm{A} \vee \mathrm{C}$, from which, by rule d_3, we have (h'$_2$): $\Gamma, \mathrm{A} \vee \mathrm{B} \vdash_{\mathscr{B}_\vee} \mathrm{C} \vee \mathrm{A}$. From ($\mathrm{h}'_1$), by (M), one gets $\Gamma, \mathrm{A} \vee \mathrm{B}, \mathrm{C} \vee \mathrm{A} \vdash_{\mathscr{B}_\vee} \mathrm{C}$, which, together with ($\mathrm{h}'_2$), results in $\Gamma, \mathrm{A} \vee \mathrm{B} \vdash_{\mathscr{B}_\vee} \mathrm{C}$ by (T). □

Theorem 4.5.6. *The calculus* $\mathscr{B}_\vee$ *is complete with respect to the matrix* $\mathbb{2}_\vee$*.*

Proof. Following the recipe presented in Section 2.7, suppose that $\Gamma \nvdash_{\mathscr{B}_\vee} \mathrm{Z}$, where $\Gamma \cup \{\mathrm{Z}\} \subseteq L_\vee$, and consider a Z-maximal $\Gamma^+ \supseteq \Gamma$ via the Lindenbaum-Asser Lemma. An specialization of property (#) for disjunction gives

$$\mathrm{A} \vee \mathrm{B} \in \Gamma^+ \text{ iff } \mathrm{A} \in \Gamma^+ \text{ or } \mathrm{B} \in \Gamma^+, \tag{$\vee$}$$

for all $\mathrm{A}, \mathrm{B} \in L_\vee$, and proving it gives the desired completeness result since $\vee$ is the sole connective in the fragment under discussion. For the right-to-left direction, assume that (a) $\mathrm{A} \in \Gamma^+$ or (b) $\mathrm{B} \in \Gamma^+$. By cases, if (a) holds, then $\Gamma^+ \vdash_{\mathscr{B}_\vee} \mathrm{A} \vee \mathrm{B}$ by (T) applied to (a) and the corresponding instance of d_1, meaning that $\mathrm{A} \vee \mathrm{B} \in \Gamma^+$, since Γ^+ is deductively closed. Similarly, the same conclusion is reached when (b) is the case, the only difference being the usage of an instance of the rule d_1' instead of d_1. For the left-to-right direction, suppose that $\mathrm{A} \vee \mathrm{B} \in \Gamma^+$. By Corollary 2.6.1.1, this implies that $\Gamma^+, \mathrm{A} \vee \mathrm{B} \nvdash_{\mathscr{B}_\vee} \mathrm{Z}$. By the contrapositive version of $\delta_\vee$ (see Lemma 4.5.5), $\Gamma^+, \mathrm{A} \nvdash_{\mathscr{B}_\vee} \mathrm{Z}$ or $\Gamma^+, \mathrm{B} \nvdash_{\mathscr{B}_\vee} \varphi$, which, again by Corollary 2.6.1.1, implies that $\mathrm{A} \in \Gamma^+$ or $\mathrm{B} \in \Gamma^+$. □

Remark 4.5.1. Notice that a sufficient condition for the preservation of the property $m_\vee$, and thus the deduction theorem $\delta_\vee$ and the completeness property ($\vee$), in any expansion of the calculus $\mathscr{B}_\vee$ by non-nullary rules is that, for any of the new rules, say r, its lifted version $\mathsf{r}^\vee$ is derivable in the expanded calculus.

A last fact about the calculus $\mathscr{B}_\vee$ is necessary for proving an important result in Section 4.7. In what follows, let $\mathsf{d}^{\vee,n}$, $n > 1$, denote the rule resulting from n successive $\vee$-liftings of rule d.

Lemma 4.5.7. *Let $\mathscr{B}_{\vee,\{f_i\}_{i\in I}}$ be any expansion of $\mathscr{B}_\vee$. If the rule*

$$\frac{\mathrm{C} \vee \mathrm{A}}{\mathrm{C} \vee \mathrm{B}}\ \mathsf{r}^\vee$$

is derivable in this calculus, then the rules

$$\frac{\mathrm{A}}{\mathrm{B}}\ \mathsf{r} \qquad \frac{\mathrm{D} \vee (\mathrm{C} \vee \mathrm{A})}{\mathrm{D} \vee (\mathrm{C} \vee \mathrm{B})}\ \mathsf{r}^{\vee,2}$$

are also derivable.

Proof. Suppose that the rule $\mathsf{r}^\vee$, as presented in the statement, is derivable in $\mathscr{B}_\vee$. Then the following derivation proves the derivability of r:

(1)	A	Assumption
(2)	$A \vee B$	1 d_1
(3)	$B \vee A$	2 d_3
(4)	$B \vee B$	3 $\mathsf{r}^{\vee}$
(5)	B	4 d_2

And the derivation below shows the derivability of $\mathsf{r}^{\vee,2}$:

(1)	$D \vee (C \vee A)$	Assumption
(2)	$(D \vee C) \vee A$	1 d_4
(3)	$(D \vee C) \vee B$	2 $\mathsf{r}^{\vee}$
(4)	$D \vee (C \vee B)$	3 d'_4

□

The expansion $\mathcal{B}_{\vee,\bot}$ is easily seen to be axiomatized by the calculus below, given Corollary 2.8.4.1.

Hilbert Calculus 14. $\mathscr{B}_{\vee,\top}$

$$\mathscr{B}_{\vee} \quad \mathscr{B}_{\top}$$

However, the calculus for the expansion of $\mathcal{B}_{\vee}$ by $\bot$ does not consist of simply adding rule b_1 to $\mathscr{B}_{\vee}$, as in the case of $\mathcal{B}_{\wedge,\bot}$. In fact, doing so would result in a still incomplete calculus, because the rule $(\mathsf{db}_1)\, A \vee \bot / A$ would be independent (consider the matrix over $\{0, 1, 2\}$ given by $\mathbb{M} = \langle \mathbf{M}, \{1\}\rangle$ such that $x \vee^{\mathbf{M}} y = 1$ if $x \neq y$ and $x \vee^{\mathbf{M}} x = x$, and $\bot^{\mathbf{M}} = 0$). Therefore, if we want to expand the calculus for disjunction, we must find another rule to produce a calculus that preserves the completeness property for disjunction and allows to prove such property for $\bot$. It turns out that rule db_1 is the one we need:

Hilbert Calculus 15. $\mathscr{B}_{\vee,\bot}$

$$\mathscr{B}_{\vee} \qquad \frac{A \vee \bot}{A}\ \mathsf{db}_1$$

Theorem 4.5.8. *The calculus $\mathscr{B}_{\vee,\bot}$ is sound with respect to the matrix $\mathcal{2}_{\vee,\bot}$.*

Proof. Soundness need to be checked only for db_1, since the rules of $\mathscr{B}_\vee$ only involve the connective $\vee$ and the soundness of its "pure" rules was already proved. Notice that, if A is evaluated to 0, the sole premiss $A \vee \bot$ would necessarily be evaluated also to 0, thus db_1 is sound with respect to $2_{\vee,\bot}$. ☐

Lemma 4.5.9. *The following rules are derivable in $\mathscr{B}_{\vee,\bot}$:*

$$\frac{B \vee (A \vee \bot)}{B \vee A}\ \mathsf{db}_1^\vee$$

$$\frac{\bot}{A}\ \mathsf{b}_1$$

Proof. The formally verified derivation of each rule is presented below, following what was explained in Chapter 3.

- $\mathsf{db}_1^\vee$

```
theorem db₁_or {a b : Prop} (h₁ : or b (or a bot)) : or b a :=
  have h₂ : or (or b a) bot, from or.d₄ h₁,
  show or b a, from db₁ h₂
```

- b_1

```
theorem b₁ {a : Prop} (h₁ : bot) : a :=
  have h₂ : or bot a, from or.d₁ h₁,
  have h₃ : or a bot, from or.d₃ h₂,
  show a, from db₁ h₃
```

☐

Theorem 4.5.10. *The calculus $\mathscr{B}_{\vee,\bot}$ is complete with respect to the matrix $2_{\vee,\bot}$.*

Proof. Because $\mathsf{db}_1^\vee$ and b_1 are derivable, properties ($\vee$) and ($\bot$) hold in $\mathscr{B}_{\vee,\bot}$, thus implying the completeness of this calculus. ☐

As usual, the expansion $\mathcal{B}_{\vee,\bot,\top}$ is axiomatized by the calculus below, given Corollary 2.8.4.1.

Hilbert Calculus 16. $\mathscr{B}_{\vee,\bot,\top}$

$$\mathscr{B}_{\vee,\perp} \quad \mathscr{B}_{\top}$$

4.5.2 Monotonic expansions of $\mathcal{B}_{\vee}$

We focus now on a theorem useful for proving the axiomatizability of monotonic expansions of $\mathcal{B}_{\vee}$. In this work, it was applied to prove the axiomatizability of the fragment $\mathcal{B}_{\mathsf{ak}}$ (see Section 4.6). We start by proving a Conjunctive Normal Form for monotonic functions over $\{0,1\}$, then we proceed to the main result.

Lemma 4.5.11. *If f^{2} is an m-ary monotonic operation over $\{0,1\}$, then there are $n \in \omega$ and $\mathrm{P}_i \in L_{\vee,\top,\perp}^{\mathrm{p}_1,\ldots,\mathrm{p}_m}$, for $1 \leq i \leq n$, such that*

$$f^{2} = \left(\bigwedge_{1\leq i\leq n} \mathrm{P}_i\right)^{2_{\vee,\wedge}}.$$

Proof. The proof goes by induction on the arity m. If $m = 0$, then f^{2} is either 1 or 0, then take $n = 1$ and $\mathrm{P}_1 = \top$ in the first case, and $\mathrm{P}_1 = \perp$ in the other. Now, suppose that the statement holds for every $m < k$, with $k > 0$, and define the operations ${f_0}^{2}$ and ${f_1}^{2}$ such that ${f_0}^{2}(x_1,\ldots,x_{k-1}) = f^{2}(x_1,\ldots,x_{k-1},0)$ and ${f_1}^{2}(x_1,\ldots,x_{k-1}) = f^{2}(x_1,\ldots,x_{k-1},1)$, which inherit from f^{2} the property of being monotonic (see Example 2.1.2). Let $\vec{x}$ abbreviate the sequence $x_1,\ldots,x_{k-1}$ and read $\vee$ and $\wedge$ as $\vee^{2}$ and $\wedge^{2}$, respectively, to simplify notation. Then we can show that (a): $f^{2}(\vec{x},x_k) = (x_k \wedge {f_1}^{2}(\vec{x})) \vee {f_0}^{2}(\vec{x})$, by analysing the possible values for x_k. In case of value 0, we have $(0 \wedge {f_1}^{2}(\vec{x})) \vee {f_0}^{2}(\vec{x}) = {f_0}^{2}(\vec{x}) = f^{2}(\vec{x},0) = f^{2}(\vec{x},x_k)$. In case of $x_k = 1$, we have $(1 \wedge {f_1}^{2}(\vec{x})) \vee {f_0}^{2}(\vec{x}) = {f_1}^{2}(\vec{x}) \vee {f_0}^{2}(\vec{x})$. Since $\langle\vec{x},0\rangle \leq \langle\vec{x},1\rangle$ and f^{2} is monotonic, we have ${f_0}^{2}(\vec{x}) \leq {f_1}^{2}(\vec{x})$, implying two cases: either ${f_0}^{2}(\vec{x}) = {f_1}^{2}(\vec{x})$ or ${f_0}^{2}(\vec{x}) = 0$ and ${f_1}^{2}(\vec{x}) = 1$. In both cases, ${f_1}^{2}(\vec{x}) \vee {f_0}^{2}(\vec{x}) = {f_1}^{2}(\vec{x}) = f^{2}(\vec{x},1) = f^{2}(\vec{x},x_k)$. From (a), by using distributivity of $\vee$ over $\wedge$, and the induction hypothesis applied to ${f_0}^{2}$ and ${f_1}^{2}$, we reach the desired result. □

Theorem 4.5.12. *Any monotonic expansion of $\mathcal{B}_{\vee}$ is axiomatizable.*

Proof. Let f be an m-ary symbol, whose 2-valued interpretation is given by f^{2}, a monotonic function over $\{0,1\}$. Notice that the cases $f^{2} = \top^{2}$ and $f^{2} = \perp^{2}$ were already axiomatized, so we may consider $m \geq 1$. By the Conjunctive Normal Form for monotonic functions presented in Lemma 4.5.11, there are $n \in \omega$ and formulas $\mathrm{P}_i \in L_{\vee}^{\mathrm{p}_1,\ldots,\mathrm{p}_m}$,

$1 \leq i \leq n$, such that $f^{\mathbb{2}} = \left(\bigwedge_{1 \leq i \leq n} \mathrm{P}_i\right)^{\mathbb{2}_{\vee,\wedge}}$. Then consider the rules

$$\frac{\mathrm{P}_1 \quad \dots \quad \mathrm{P}_m}{f(\mathrm{p}_1, \dots, \mathrm{p_m})}\ \mathsf{f}_0 \qquad \frac{f(\mathrm{p}_1, \dots, \mathrm{p_m})}{\mathrm{P}_i}\ \mathsf{f}_\mathsf{i}, 1 \leq i \leq n$$

and the following calculus:

Hilbert Calculus 17. $\mathscr{B}_{\vee,f}$, f monotonic

$$\mathscr{B}_\vee$$

$$\frac{\mathrm{p}_{m+1} \vee \mathrm{P}_1 \quad \dots \quad \mathrm{p}_{m+1} \vee \mathrm{P}_m}{\mathrm{p}_{m+1} \vee f(\mathrm{p}_1, \dots, \mathrm{p_m})}\ \mathsf{f}_0^\vee$$

$$\frac{\mathrm{p}_{m+1} \vee f(\mathrm{p}_1, \dots, \mathrm{p_m})}{\mathrm{p}_{m+1} \vee \mathrm{P}_i}\ \mathsf{f}_\mathsf{i}^\vee, 1 \leq i \leq n$$

We proceed to argue about the soundness of $\mathscr{B}_{\vee,f}$ with respect to $\mathbb{2}_{\vee,f}$. Notice that the rules of $\mathscr{B}_\vee$ were already proved sound earlier in this section. About rule $\mathsf{f}_0^\vee$, if its conclusion is evaluated to false, then p_{m+1} and P_i are both evaluated to false, and, since $f(\mathrm{p}_1, \dots, \mathrm{p}_m)$ is evaluated to a conjunction of all P_i, it is also falsified, so there is no way to evaluate the premiss to true. The argument for the other rules is analogous.

To prove the completeness of this calculus, first notice that the completeness property for f, as a direct consequence of the Conjunctive Normal Form referred above, is given by

$$f(\mathrm{A}_1, \dots, \mathrm{A}_n) \in \Gamma^+ \text{ iff } \mathrm{P}_1^\sigma, \dots, \mathrm{P}_n^\sigma \in \Gamma^+ \tag{f}$$

where σ is a substitution such that $\sigma(\mathrm{p}_i) = \mathrm{A}_i$, for all $1 \leq i \leq n$. The desired result hence follows by showing that both $(\vee)$ and (f) hold in $\mathscr{B}_{\vee,f}$. By Lemma 4.5.7, $\mathsf{f}_\mathsf{i}^{\vee,2}$ holds for every rule $\mathsf{f}_\mathsf{i}^\vee$, where $0 \leq i \leq n$, and hence $m_\vee$ and $\delta_\vee$ hold, implying finally that $(\vee)$ also holds in this calculus. Now, it remains to show that (f) also holds. For that, consider the fact that the rules f_i, where $1 \leq i \leq n$, hold in this calculus, by Lemma 4.5.7. Then from the left to the right, apply the instances of rules f_i, where $1 \leq i \leq n$, corresponding to the appropriate substitution σ. From the right to the left, use the appropriate instance of rule f_0. □

4.6 $\mathcal{B}_{\mathsf{ak}}$, $\mathcal{B}_{\mathsf{ak},\top}$

The classical connective ak may be defined from those in $\mathcal{B}$ via the translation $\mathbf{t}(\mathsf{ak}) = \lambda p, q, r. p \vee (q \wedge r)$, hence its interpretation in $\mathbb{2}_{\mathsf{ak}}$ is monotonic. Moreover, notice that $\vee^{\mathbb{2}}$ is definable by $\lambda x, y.\mathsf{ak}^{\mathbb{2}}(x, y, y)$, thus $\mathbb{2}_{\mathsf{ak}}$ is a monotonic expansion of $\mathbb{2}_{\vee}$. Henceforth, we will use the abbreviation $A \vee^{\mathsf{ak}} B$ for $\mathsf{ak}(A, B, B)$. Then, by Theorem 4.5.12, the following calculus axiomatizes the fragment $\mathcal{B}_{\mathsf{ak}}$:

Hilbert Calculus 18. $\mathscr{B}_{\mathsf{ak}}$

$$\mathscr{B}_{\vee}, \text{ with } \vee := \vee^{\mathsf{ak}}$$

$$\frac{D \vee^{\mathsf{ak}} (A \vee^{\mathsf{ak}} B) \quad D \vee^{\mathsf{ak}} (A \vee^{\mathsf{ak}} C)}{D \vee^{\mathsf{ak}} \mathsf{ak}(A, B, C)}\ \mathsf{ak}_1 \qquad \frac{D \vee^{\mathsf{ak}} \mathsf{ak}(A, B, C)}{D \vee^{\mathsf{ak}} (A \vee^{\mathsf{ak}} B)}\ \mathsf{ak}_2 \qquad \frac{D \vee^{\mathsf{ak}} \mathsf{ak}(A, B, C)}{D \vee^{\mathsf{ak}} (A \vee^{\mathsf{ak}} C)}\ \mathsf{ak}_3$$

Then, by Corollary 2.8.4.1, the following calculus axiomatizes $\mathcal{B}_{\mathsf{ak},\top}$:

Hilbert Calculus 19. $\mathscr{B}_{\mathsf{ak},\top}$

$$\mathscr{B}_{\mathsf{ak}} \quad \mathscr{B}_{\top}$$

4.7 $\mathcal{B}_{\mathsf{ad}}$, $\mathcal{B}_{\mathsf{ad},\top}$

The classical connective ad may be defined from those in $\mathcal{B}$ by means of the translation $\mathbf{t}(\mathsf{ad}) = \lambda p, q, r. p \vee (q \wedge \neg r)$. In what follows, abbreviate $\mathsf{ad}(A, B, A)$ by $A \vee^{\mathsf{ad}} B$ and notice that its semantics in $\mathbb{2}_{\mathsf{ad}}$ is the same as $\vee$ in $\mathbb{2}$. Although the proposed calculus is large, with twenty-five rules, the way it was conceived is not hard to understand. We establish the rules $\mathsf{ad}_1 - \mathsf{ad}_{11}$, then we take from $\mathscr{B}_{\vee}$ the rules $\mathsf{d}_2 - \mathsf{d}_4$, and finally add the $\vee^{\mathsf{ad}}$-lifted (same notion of $\vee$-lifted rules) versions of rules $\mathsf{ad}_1 - \mathsf{ad}_{11}$. Such choices will greatly simplify the proofs that are to come. Below we present $\mathscr{B}_{\mathsf{ad}}$ followed by the proof of soundness with respect to $\mathbb{2}_{\mathsf{ad}}$.

Hilbert Calculus 20. $\mathscr{B}_{\mathsf{ad}}$

$$\frac{C \quad \mathsf{ad}(A, B, C)}{A}\ \mathsf{ad}_1$$

$$\frac{A}{\mathsf{ad}(\mathsf{ad}(C, A, B), A, C)}\ \mathsf{ad}_2$$

$$\frac{\mathsf{ad}(A, B, C)}{\mathsf{ad}(\mathsf{ad}(\mathsf{ad}(\mathsf{ad}(F, A, D), A, \mathsf{ad}(E, A, D)), A, \mathsf{ad}(\mathsf{ad}(F, A, E), A, D)), B, C)}\ \mathsf{ad}_3$$

$$\frac{\mathsf{ad}(A, B, C)}{\mathsf{ad}(\mathsf{ad}(D, A, \mathsf{ad}(D, A, \mathsf{ad}(E, A, D))), B, C)}\ \mathsf{ad}_4$$

$$\frac{\mathsf{ad}(A, B, C)}{B \vee^{\mathsf{ad}} A}\ \mathsf{ad}_5$$

$$\frac{A}{\mathsf{ad}(A, B, C)}\ \mathsf{ad}_6$$

$$\frac{\mathsf{ad}(C \vee^{\mathsf{ad}} D, A, B)}{\mathsf{ad}(C, A, B) \vee^{\mathsf{ad}} \mathsf{ad}(D, A, B)}\ \mathsf{ad}_7$$

$$\frac{\mathsf{ad}(C, A, B) \vee^{\mathsf{ad}} \mathsf{ad}(D, A, B)}{\mathsf{ad}(C \vee^{\mathsf{ad}} D, A, B)}\ \mathsf{ad}_8$$

$$\frac{\mathsf{ad}(A, B, C) \quad \mathsf{ad}(D, E, A)}{\mathsf{ad}(D, B, C)}\ \mathsf{ad}_9$$

$$\frac{\mathsf{ad}(\mathsf{ad}(E, D, C), A, B)}{\mathsf{ad}(E, A, B) \vee^{\mathsf{ad}} \mathsf{ad}(E, D, C)}\ \mathsf{ad}_{10}$$

$$\frac{\mathsf{ad}(E, A, B) \vee^{\mathsf{ad}} \mathsf{ad}(E, D, C)}{\mathsf{ad}(\mathsf{ad}(E, D, C), A, B)}\ \mathsf{ad}_{11}$$

$$\frac{A \vee^{\mathsf{ad}} A}{A}\ \mathsf{ad}_{12}$$

$$\frac{A \vee^{\mathsf{ad}} B}{B \vee^{\mathsf{ad}} A}\ \mathsf{ad}_{13}$$

$$\frac{A \vee^{\mathsf{ad}} (B \vee^{\mathsf{ad}} C)}{(A \vee^{\mathsf{ad}} B) \vee^{\mathsf{ad}} C}\ \mathsf{ad}_{14}$$

$$\frac{D \vee^{\mathsf{ad}} C \quad D \vee^{\mathsf{ad}} \mathsf{ad}(A, B, C)}{D \vee^{\mathsf{ad}} A}\ \mathsf{ad}_{15}$$

$$\frac{D \vee^{\mathsf{ad}} A}{D \vee^{\mathsf{ad}} \mathsf{ad}(\mathsf{ad}(C, A, B), A, C)}\ \mathsf{ad}_{16}$$

$$\frac{\mathrm{G} \vee^{\mathsf{ad}} \mathsf{ad}(\mathrm{A},\mathrm{B},\mathrm{C})}{\mathrm{G} \vee^{\mathsf{ad}} \mathsf{ad}(\mathsf{ad}(\mathsf{ad}(\mathsf{ad}(\mathrm{F},\mathrm{A},\mathrm{D}),\mathrm{A},\mathsf{ad}(\mathrm{E},\mathrm{A},\mathrm{D})),\mathrm{A},\mathsf{ad}(\mathsf{ad}(\mathrm{F},\mathrm{A},\mathrm{E}),\mathrm{A},\mathrm{D})),\mathrm{B},\mathrm{C})}\ \mathsf{ad}_{17}$$

$$\frac{\mathrm{F} \vee^{\mathsf{ad}} \mathsf{ad}(\mathrm{A},\mathrm{B},\mathrm{C})}{\mathrm{F} \vee^{\mathsf{ad}} \mathsf{ad}(\mathsf{ad}(\mathrm{D},\mathrm{A},\mathsf{ad}(\mathrm{D},\mathrm{A},\mathsf{ad}(\mathrm{E},\mathrm{A},\mathrm{D}))),\mathrm{B},\mathrm{C})}\ \mathsf{ad}_{18}$$

$$\frac{\mathrm{D} \vee^{\mathsf{ad}} \mathsf{ad}(\mathrm{A},\mathrm{B},\mathrm{C})}{\mathrm{D} \vee^{\mathsf{ad}} \mathsf{ad}(\mathrm{B},\mathrm{A},\mathrm{B})}\ \mathsf{ad}_{19}$$

$$\frac{\mathrm{D} \vee^{\mathsf{ad}} \mathrm{A}}{\mathrm{D} \vee^{\mathsf{ad}} \mathsf{ad}(\mathrm{A},\mathrm{B},\mathrm{C})}\ \mathsf{ad}_{20}$$

$$\frac{\mathrm{E} \vee^{\mathsf{ad}} \mathsf{ad}(\mathrm{C} \vee^{\mathsf{ad}} \mathrm{D},\mathrm{A},\mathrm{B})}{\mathrm{E} \vee^{\mathsf{ad}} (\mathsf{ad}(\mathrm{C},\mathrm{A},\mathrm{B}) \vee^{\mathsf{ad}} \mathsf{ad}(\mathrm{D},\mathrm{A},\mathrm{B}))}\ \mathsf{ad}_{21}$$

$$\frac{\mathrm{E} \vee^{\mathsf{ad}} (\mathsf{ad}(\mathrm{C},\mathrm{A},\mathrm{B}) \vee^{\mathsf{ad}} \mathsf{ad}(\mathrm{D},\mathrm{A},\mathrm{B}))}{\mathrm{E} \vee^{\mathsf{ad}} \mathsf{ad}(\mathrm{C} \vee^{\mathsf{ad}} \mathrm{D},\mathrm{A},\mathrm{B})}\ \mathsf{ad}_{22}$$

$$\frac{\mathrm{F} \vee^{\mathsf{ad}} \mathsf{ad}(\mathrm{B},\mathrm{D},\mathrm{E}) \quad \mathrm{F} \vee^{\mathsf{ad}} \mathsf{ad}(\mathrm{C},\mathrm{A},\mathrm{B})}{\mathrm{F} \vee^{\mathsf{ad}} \mathsf{ad}(\mathrm{E},\mathrm{D},\mathrm{C})}\ \mathsf{ad}_{23}$$

$$\frac{\mathrm{F} \vee^{\mathsf{ad}} \mathsf{ad}(\mathsf{ad}(\mathrm{E},\mathrm{D},\mathrm{C}),\mathrm{A},\mathrm{B})}{\mathrm{F} \vee^{\mathsf{ad}} (\mathsf{ad}(\mathrm{E},\mathrm{A},\mathrm{B}) \vee^{\mathsf{ad}} \mathsf{ad}(\mathrm{E},\mathrm{D},\mathrm{C}))}\ \mathsf{ad}_{24}$$

$$\frac{\mathrm{F} \vee^{\mathsf{ad}} (\mathsf{ad}(\mathrm{E},\mathrm{A},\mathrm{B}) \vee^{\mathsf{ad}} \mathsf{ad}(\mathrm{E},\mathrm{D},\mathrm{C}))}{\mathrm{F} \vee^{\mathsf{ad}} \mathsf{ad}(\mathsf{ad}(\mathrm{E},\mathrm{D},\mathrm{C}),\mathrm{A},\mathrm{B})}\ \mathsf{ad}_{25}$$

Theorem 4.7.1. *The calculus $\mathscr{B}_{\mathsf{ad}}$ is sound with respect to the matrix $\mathbb{2}_{\mathsf{ad}}$.*

Proof. Let v be a $\mathbb{2}_{\mathsf{ad}}$-valuation, and, to simplify notation, denote also by v the $\mathbb{2}$-valuation v' such that $v = \mathbf{t} \circ v'$. For rule ad_1, suppose that $v(\mathrm{C}) = 1$ and $v(\mathsf{ad}(\mathrm{A},\mathrm{B},\mathrm{C})) = 1$. Then, $v(\mathrm{B} \wedge \neg\mathrm{C}) = 0$, so $v(\mathrm{A}) = 1$. For rule ad_2, suppose that $v(\mathrm{A}) = 1$. Then any value of $v(\mathrm{C})$ leads v to assign 1 to the conclusion. For rule ad_3, suppose that $v(\mathsf{ad}(\mathrm{A},\mathrm{B},\mathrm{C})) = 1$. Then, we have two cases:

- $v(\mathrm{B} \wedge \neg\mathrm{C}) = 1$: clearly causes v to assign 1 to the premiss; or
- $v(\mathrm{A}) = 1$: our goal is then to show (a): $v(\mathsf{ad}(\mathsf{ad}(\mathrm{F},\mathrm{A},\mathrm{D}),\mathrm{A},\mathsf{ad}(\mathrm{E},\mathrm{A},\mathrm{D}))) = 1$ or (b): $v(\mathsf{ad}(\mathsf{ad}(\mathrm{F},\mathrm{A},\mathrm{E}),\mathrm{A},\mathrm{D})) = 0$. Let us work on the possible values for D, E and F under v:
  - $v(\mathrm{D}) = 0$ or $v(\mathrm{F}) = 1$: we have $v(\mathsf{ad}(\mathrm{F},\mathrm{A},\mathrm{D})) = 1$, thus (a).
  - $v(\mathrm{D}) = 1$, $v(\mathrm{E}) = 1$ and $v(\mathrm{F}) = 0$: we have $v(\mathsf{ad}(\mathrm{F},\mathrm{A},\mathrm{E})) = 0$, thus (b).
  - $v(\mathrm{D}) = 1$ and $v(\mathrm{E}) = 0$: we have $v(\mathsf{ad}(\mathrm{E},\mathrm{A},\mathrm{D})) = 0$, thus (a).

For rule ad_4, the proof is similar, with the non-trivial case being $v(\mathrm{A}) = 1$. Our goal is then to show that (c): $v(\mathsf{ad}(\mathrm{D}, \mathrm{A}, \mathsf{ad}(\mathrm{D}, \mathrm{A}, \mathsf{ad}(\mathrm{E}, \mathrm{A}, \mathrm{D})))) = 1$. Notice that if $v(\mathrm{D}) = 1$, then (c). Otherwise, we have $v(\mathsf{ad}(\mathrm{E}, \mathrm{A}, \mathrm{D})) = 1$, then $v(\mathsf{ad}(\mathrm{D}, \mathrm{A}, \mathsf{ad}(\mathrm{E}, \mathrm{A}, \mathrm{D}))) = 0$, and thus (c). The soundness of rules ad_5 and ad_6 follows directly from the determinants of ad in $\mathbb{2}_{\mathsf{ad}}$. For rule ad_7, suppose that $v(\mathsf{ad}(\mathrm{C} \vee^{\mathsf{ad}} \mathrm{D}, \mathrm{A}, \mathrm{B})) = 1$. Then (d): $v(\mathrm{C} \vee^{\mathsf{ad}} \mathrm{D}) = 1$ or (e): $v(\mathrm{A}) = 1$ and $\mathrm{B} = 0$. In the first case, either C or D is assigned the value 1, so $v(\mathsf{ad}(\mathrm{C}, \mathrm{A}, \mathrm{B})) = 1$ or $v(\mathsf{ad}(\mathrm{D}, \mathrm{A}, \mathrm{B})) = 1$, and v assigns 1 to the conclusion. In the other case, we have $v(\mathsf{ad}(\mathrm{C}, \mathrm{A}, \mathrm{B})) = 1$ and $v(\mathsf{ad}(\mathrm{D}, \mathrm{A}, \mathrm{B})) = 1$, and the conclusion also gets the value 1 under v. For rule ad_8, suppose that $v(\mathsf{ad}(\mathrm{C} \vee^{\mathsf{ad}} \mathrm{D}, \mathrm{A}, \mathrm{B})) = 0$. Then $v(\mathrm{C} \vee^{\mathsf{ad}} \mathrm{D}) = 0$, i.e. $v(\mathrm{C}) = 0$ and $v(\mathrm{D}) = 0$, and $v(\mathrm{A}) = 0$ or $v(\mathrm{B}) = 1$. Notice that, under these conditions, $v(\mathsf{ad}(\mathrm{C}, \mathrm{A}, \mathrm{B})) = 0$ and $v(\mathsf{ad}(\mathrm{D}, \mathrm{A}, \mathrm{B})) = 0$, so v assigns 0 to the premiss. For rule ad_9, suppose that $v(\mathsf{ad}(\mathrm{D}, \mathrm{B}, \mathrm{C})) = 0$, so $v(\mathrm{D}) = 0$ and $v(\mathrm{B}) = 0$ or $v(\mathrm{C}) = 1$. If $v(\mathrm{B}) = 0$, case $v(\mathrm{A}) = 0$, we have $v(\mathsf{ad}(\mathrm{A}, \mathrm{B}, \mathrm{C})) = 0$ and, case $v(\mathrm{A}) = 1$, we have $v(\mathsf{ad}(\mathrm{D}, \mathrm{E}, \mathrm{A})) = 0$. In the other hand, if $v(\mathrm{C}) = 1$, when $v(\mathrm{A}) = 0$, we have $v(\mathsf{ad}(\mathrm{A}, \mathrm{B}, \mathrm{C})) = 0$, and when $v(\mathrm{A}) = 1$, we have $v(\mathsf{ad}(\mathrm{D}, \mathrm{E}, \mathrm{A})) = 0$. For rule ad_{10}, suppose that $v(\mathsf{ad}(\mathsf{ad}(\mathrm{E}, \mathrm{D}, \mathrm{C}), \mathrm{A}, \mathrm{B})) = 1$, so $v(\mathsf{ad}(\mathrm{E}, \mathrm{D}, \mathrm{C})) = 1$ or $v(\mathrm{A}) = 1$ and $v(\mathrm{B}) = 0$. In the first case, clearly v assigns 1 to the conclusion. In the second, we have $v(\mathsf{ad}(\mathrm{E}, \mathrm{A}, \mathrm{B})) = 1$, so the conclusion also is assigned to 1. The proof for ad_{11} is very similar. For rules ad_{12}, ad_{13} and ad_{14}, use Theorem 4.5.1. For the remaining rules, it is enough to show that if a rule is sound with respect to $\mathbb{2}$, then its $\vee$-lifted version is also sound with respect to $\mathbb{2}$. So suppose that $(\mathsf{r})\, \mathrm{A}_1, \ldots, \mathrm{A}_n/\mathrm{A}_{n+1}$ is sound with respect to $\mathbb{2}$ and consider its $\vee$-lifted version given by $(\mathsf{r}^{\vee})\, \mathrm{B}\vee\mathrm{A}_1, \ldots, \mathrm{B}\vee\mathrm{A}_n/\mathrm{B}\vee\mathrm{A}_{n+1}$. Then assume that $v(\mathrm{B}\vee\mathrm{A}_1) = \cdots = v(\mathrm{B}\vee\mathrm{A}_n) = 1$, hence either $v(\mathrm{B}) = 1$ or $v(\mathrm{A}_1) = \cdots = v(\mathrm{A}_n) = 1$. In the first case, trivially $v(\mathrm{B} \vee \mathrm{A}_{n+1}) = 1$, and, in the second, since r is sound, we have $v(\mathrm{A}_{n+1}) = 1$, thus $v(\mathrm{B} \vee \mathrm{A}_{n+1}) = 1$. □

Lemma 4.7.2. *Where* $\vee := \vee^{\mathsf{ad}}$, *properties* $m_\vee$ *and* $\delta_\vee$ *(see Lemma 4.5.3 and Lemma 4.5.5, respectively) hold in* $\mathscr{B}_{\mathsf{ad}}$.

Proof. In view of Remark 4.5.1, we argue on the derivability of the $\vee$-lifted rules of $\mathscr{B}_{\mathsf{ad}}$. Notice that, by taking $\vee := \vee^{\mathsf{ad}}$, the rules $\mathsf{ad}_{12}, \mathsf{ad}_{13}$ and ad_{14} are the same as d_2, d_3 and d_4 (see Section 4.5). Also, rule d_1 is a special case of ad_6 (just take $\mathrm{C} = \mathrm{A}$), called ad_6' here. By Lemma 4.5.2, the presence of those rules implies that their $\vee$-lifted versions are derivable. Moreover, rules $\mathsf{ad}_{15}, \ldots, \mathsf{ad}_{25}$ are the $\vee$-lifted versions of rules $\mathsf{ad}_1, \ldots, \mathsf{ad}_{11}$. Finally, by Lemma 4.5.7, the lifted versions of $\mathsf{ad}_{15}, \ldots, \mathsf{ad}_{25}$ are derivable in this calculus. □

In what follows, if r is an n-ary rule, with $n \in \omega$, let r^{ad}, the ad-lifted version of r, be the rule given by the set of instances $\langle \mathsf{ad}(\mathrm{A}_1, \mathrm{C}, \mathrm{D}), \ldots, \mathsf{ad}(\mathrm{A}_n, \mathrm{C}, \mathrm{D}), \mathsf{ad}(\mathrm{B}, \mathrm{C}, \mathrm{D}) \rangle$, where

$\langle \mathrm{A}_1, \ldots, \mathrm{A}_n, \mathrm{B}\rangle$ is an instance of r and $\mathrm{C}, \mathrm{D} \in L_{\mathsf{ad}}$. The main purpose of the next lemma is to show that all ad-lifted versions of the primitive rules of ad are derivable in $\mathscr{B}_{\mathsf{ad}}$, an important step towards the completeness of this calculus with respect to $\mathfrak{2}_{\mathsf{ad}}$.

Lemma 4.7.3. *The following rules are derivable in* $\mathscr{B}_{\mathsf{ad}}$*:*

$$\frac{\mathrm{A} \quad \mathrm{B}}{\mathsf{ad}(\mathrm{A}, \mathrm{B}, \mathrm{C})}\ \mathsf{ad}_{26}$$

$$\frac{\mathrm{A}}{\mathsf{ad}(\mathrm{C}, \mathrm{A}, \mathsf{ad}(\mathrm{C}, \mathrm{A}, \mathsf{ad}(\mathrm{B}, \mathrm{A}, \mathrm{C})))}\ \mathsf{ad}'_4$$

$$\frac{\mathsf{ad}(\mathsf{ad}(\mathrm{A}, \mathrm{B}, \mathrm{C}), \mathrm{D}, \mathrm{E})}{\mathsf{ad}(\mathsf{ad}(\mathrm{A}, \mathrm{D}, \mathrm{E}), \mathrm{B}, \mathrm{C})}\ \mathsf{ad}_{27}$$

$$\frac{\mathrm{A}}{\mathrm{A} \vee^{\mathsf{ad}} \mathrm{B}}\ \mathsf{ad}'_6$$

$$\frac{\mathsf{ad}(\mathrm{C}, \mathrm{D}, \mathrm{E}) \quad \mathsf{ad}(\mathsf{ad}(\mathrm{A}, \mathrm{B}, \mathrm{C}), \mathrm{D}, \mathrm{E})}{\mathsf{ad}(\mathrm{A}, \mathrm{D}, \mathrm{E})}\ \mathsf{ad}^{\mathsf{ad}}_1$$

$$\frac{\mathsf{ad}(\mathrm{A}, \mathrm{D}, \mathrm{E})}{\mathsf{ad}(\mathsf{ad}(\mathsf{ad}(\mathrm{C}, \mathrm{A}, \mathrm{B}), \mathrm{A}, \mathrm{C}), \mathrm{D}, \mathrm{E})}\ \mathsf{ad}^{\mathsf{ad}}_2$$

$$\frac{\mathsf{ad}(\mathsf{ad}(\mathrm{A}, \mathrm{B}, \mathrm{C}), \mathrm{G}, \mathrm{H})}{\mathsf{ad}(\mathsf{ad}(\mathsf{ad}(\mathsf{ad}(\mathsf{ad}(\mathrm{F}, \mathrm{A}, \mathrm{D}), \mathrm{A}, \mathsf{ad}(\mathrm{E}, \mathrm{A}, \mathrm{D})), \mathrm{A}, \mathsf{ad}(\mathsf{ad}(\mathrm{F}, \mathrm{A}, \mathrm{E}), \mathrm{A}, \mathrm{D})), \mathrm{B}, \mathrm{C}), \mathrm{G}, \mathrm{H})}\ \mathsf{ad}^{\mathsf{ad}}_3$$

$$\frac{\mathsf{ad}(\mathsf{ad}(\mathrm{A}, \mathrm{B}, \mathrm{C}), \mathrm{F}, \mathrm{G})}{\mathsf{ad}(\mathsf{ad}(\mathsf{ad}(\mathrm{D}, \mathrm{A}, \mathsf{ad}(\mathrm{D}, \mathrm{A}, \mathsf{ad}(\mathrm{E}, \mathrm{A}, \mathrm{D}))), \mathrm{B}, \mathrm{C}), \mathrm{F}, \mathrm{G})}\ \mathsf{ad}^{\mathsf{ad}}_4$$

$$\frac{\mathsf{ad}(\mathsf{ad}(\mathrm{A}, \mathrm{B}, \mathrm{C}), \mathrm{D}, \mathrm{E})}{\mathsf{ad}(\mathrm{B} \vee^{\mathsf{ad}} \mathrm{A}, \mathrm{D}, \mathrm{E})}\ \mathsf{ad}^{\mathsf{ad}}_5$$

$$\frac{\mathsf{ad}(\mathrm{A}, \mathrm{D}, \mathrm{E})}{\mathsf{ad}(\mathsf{ad}(\mathrm{A}, \mathrm{B}, \mathrm{C}), \mathrm{D}, \mathrm{E})}\ \mathsf{ad}^{\mathsf{ad}}_6$$

$$\frac{\mathsf{ad}(\mathsf{ad}(\mathrm{C} \vee^{\mathsf{ad}} \mathrm{D}, \mathrm{A}, \mathrm{B}), \mathrm{E}, \mathrm{F})}{\mathsf{ad}(\mathsf{ad}(\mathrm{C}, \mathrm{A}, \mathrm{B}) \vee^{\mathsf{ad}} \mathsf{ad}(\mathrm{D}, \mathrm{A}, \mathrm{B}), \mathrm{E}, \mathrm{F})}\ \mathsf{ad}^{\mathsf{ad}}_7$$

$$\frac{\mathsf{ad}(\mathrm{A} \vee^{\mathsf{ad}} \mathrm{A}, \mathrm{B}, \mathrm{C})}{\mathsf{ad}(\mathrm{A}, \mathrm{B}, \mathrm{C})}\ \mathsf{ad}^{\mathsf{ad}}_{12}$$

$$\frac{\mathsf{ad}(\mathrm{A} \vee^{\mathsf{ad}} \mathrm{B}, \mathrm{C}, \mathrm{D})}{\mathsf{ad}(\mathrm{B} \vee^{\mathsf{ad}} \mathrm{A}, \mathrm{C}, \mathrm{D})}\ \mathsf{ad}^{\mathsf{ad}}_{13}$$

$$\frac{\mathsf{ad}(\mathrm{A} \vee^{\mathsf{ad}} (\mathrm{B} \vee^{\mathsf{ad}} \mathrm{C}), \mathrm{D}, \mathrm{E})}{\mathsf{ad}((\mathrm{A} \vee^{\mathsf{ad}} \mathrm{B}) \vee^{\mathsf{ad}} \mathrm{C}, \mathrm{D}, \mathrm{E})}\ \mathsf{ad}^{\mathsf{ad}}_{14}$$

$$\frac{\mathsf{ad}(\mathsf{ad}(\mathrm{C}, \mathrm{A}, \mathrm{B}), \mathrm{E}, \mathrm{F})}{\mathsf{ad}(\mathsf{ad}(\mathrm{C} \vee^{\mathsf{ad}} \mathrm{D}, \mathrm{A}, \mathrm{B}), \mathrm{E}, \mathrm{F})}\ \mathsf{ad}'_8$$

$$\frac{\mathsf{ad}(\mathsf{ad}(\mathrm{C} \vee^{\mathsf{ad}} \mathrm{D}, \mathrm{A}, \mathrm{B}), \mathrm{E}, \mathrm{F})}{\mathsf{ad}(\mathsf{ad}(\mathrm{D} \vee^{\mathsf{ad}} \mathrm{C}, \mathrm{A}, \mathrm{B}), \mathrm{E}, \mathrm{F})}\ \mathsf{ad}''_8$$

$$\frac{\mathsf{ad}(\mathsf{ad}(\mathrm{C}, \mathrm{A}, \mathrm{B}) \vee^{\mathsf{ad}} \mathsf{ad}(\mathrm{D}, \mathrm{A}, \mathrm{B}), \mathrm{E}, \mathrm{F})}{\mathsf{ad}(\mathsf{ad}(\mathrm{C} \vee^{\mathsf{ad}} \mathrm{D}, \mathrm{A}, \mathrm{B}), \mathrm{E}, \mathrm{F})}\ \mathsf{ad}^{\mathsf{ad}}_8$$

$$\frac{\mathsf{ad}(\mathsf{ad}(\mathrm{A},\mathrm{B},\mathrm{C}),\mathrm{F},\mathrm{G}) \quad \mathsf{ad}(\mathsf{ad}(\mathrm{D},\mathrm{E},\mathrm{A}),\mathrm{F},\mathrm{G})}{\mathsf{ad}(\mathsf{ad}(\mathrm{D},\mathrm{B},\mathrm{C}),\mathrm{F},\mathrm{G})}\ \mathsf{ad}_9^{\mathsf{ad}}$$

$$\frac{\mathsf{ad}(\mathsf{ad}(\mathsf{ad}(\mathrm{E},\mathrm{D},\mathrm{C}),\mathrm{A},\mathrm{B}),\mathrm{F},\mathrm{G})}{\mathsf{ad}(\mathsf{ad}(\mathrm{E},\mathrm{A},\mathrm{B}) \vee^{\mathsf{ad}} \mathsf{ad}(\mathrm{E},\mathrm{D},\mathrm{C}),\mathrm{F},\mathrm{G})}\ \mathsf{ad}_{10}^{\mathsf{ad}}$$

$$\frac{\mathsf{ad}(\mathsf{ad}(\mathrm{E},\mathrm{A},\mathrm{B}) \vee^{\mathsf{ad}} \mathsf{ad}(\mathrm{E},\mathrm{D},\mathrm{C}),\mathrm{F},\mathrm{G})}{\mathsf{ad}(\mathsf{ad}(\mathsf{ad}(\mathrm{E},\mathrm{D},\mathrm{C}),\mathrm{A},\mathrm{B}),\mathrm{F},\mathrm{G})}\ \mathsf{ad}_{11}^{\mathsf{ad}}$$

Proof. The formally verified derivation of each rule is presented below, following what was explained in Chapter 3. Some derivations apply specialized versions of $\delta_\vee$ given by the sequent-style rules

$$\frac{\mathrm{A} \succ \mathrm{C} \quad \mathrm{B} \succ \mathrm{C}}{\mathrm{A} \vee \mathrm{B} \succ \mathrm{C}}\ \delta_{\vee 1} \qquad \frac{\mathrm{D}, \mathrm{A} \succ \mathrm{C} \quad \mathrm{D}, \mathrm{B} \succ \mathrm{C}}{\mathrm{D}, \mathrm{A} \vee \mathrm{B} \succ \mathrm{C}}\ \delta_{\vee 2}.$$

- ad_{26}

```
theorem ad₂₆ {a b c : Prop} (h₁ : a) (h₂ : b) : ad a b c :=
  have h₃ : ad (ad a b c) b a, from ad₂ h₂,
  show ad a b c, from ad₁ h₁ h₃
```

- ad_4'

```
theorem ad₄' {a b c : Prop} (h₁ : a) : ad c a (ad c a (ad b a c)) :=
  have h₂ : ad a a a, from ad₂₆ h₁ h₁,
  have h₃ : ad (ad c a (ad c a (ad b a c))) a a, from ad₄ h₂,
  show ad c a (ad c a (ad b a c)), from ad₁ h₁ h₃
```

- ad_{27}

```
theorem ad₂₇ {a b c d e : Prop} (h₁ : ad (ad a b c) d e) : ad (ad a d e) b c :=
  have h₂ : (ad a d e) or (ad a b c), from ad₁₀ h₁,
  have h₃ : (ad a b c) or (ad a d e), from ad₁₃ h₂,
  show ad (ad a d e) b c, from ad₁₁ h₃
```

- ad_6'

```
theorem ad₆' {a b : Prop} (h₁ : a) : a or b :=
  ad₆ h₁
```

- $\mathsf{ad}_1^{\mathsf{ad}}$

```
theorem ad₁_ad {a b c d e : Prop} (h₁ : ad c d e) (h₂ : ad (ad a b c) d e) : ad a d e :=
  have h₃ : (ad (ad a b c) d e) → ((ad a d e) or (ad a b c)), from ad₁₀,
  have h₄ : ad a d e → ad a d e, from R,
  have h₅ : ad c d e → ad a b c → ad a d e, from ad₉,
  have h₆ : ad c d e → ad a d e → ad a d e, from M₁ h₄,
  have h₇ : ad c d e → ((ad a d e) or (ad a b c)) → ad a d e, from δ_or₂ h₆ h₅,
  have h₈ : ad c d e → (ad (ad a b c) d e) → ((ad a d e) or (ad a b c)), from M₁ h₃,
  show ad a d e, from (T₂ h₈ h₇) h₁ h₂
```

- $\mathsf{ad}_2^{\mathsf{ad}}$

```
theorem ad₂_ad {a b c d e : Prop} (h₁ : ad a d e) : ad (ad (ad c a b) a c) d e :=
  let b' := ad c a b in
    have h₂ : (ad a d e) → (((ad c a c) or b') or ((ad c d e) or b')),
      from (assume h, ad₂₀ $ ad₁₃ $ ad₂₀ $ ad₁₀ $ ad₈ $ ad₁₃ $ ad₆' h),
    have h₃ : ((ad c a c) or b') → ((ad b' d e) or (ad b' a c)),
      from (assume h, ad₁₃ $ ad₆' $ ad₁₁ h),
    have h₄ : ((ad c d e) or b') → ((ad b' d e) or (ad b' a c)),
      from (assume h, ad₆' $ ad₁₁ h),
    have h₅ : (((ad c a c) or b') or ((ad c d e) or b')) → ((ad b' d e) or (ad b' a c)),
      from δ_or₁ h₃ h₄,
    have h₆ : ad a d e → ((ad b' d e) or (ad b' a c)),
      from T₁ h₂ h₅,
    show ad (ad b' a c) d e,
      from ad₁₁ (h₆ h₁)
```

- $\mathsf{ad}_3^{\mathsf{ad}}$

```
theorem ad₃_ad {a b c d e f g h : Prop} (h₁ : ad (ad a b c) g h) :
  ad (ad (ad (ad (ad f a d) a (ad e a d)) a (ad (ad f a e) a d)) b c) g h :=
  let j := ad (ad f a d) a (ad e a d), k := ad (ad f a e) a d, i := ad j a k in
    have h₂ : (ad (ad a b c) g h) → ((ad a g h) or (ad a b c)),
      from ad₁₀,
    have h₃ : ad a g h → ad (ad i b c) g h,
      from (assume h₃₁, ad₁₁ $ ad₆' $ ad₃ h₃₁),
    have h₄ : ad a b c → ad (ad i b c) g h,
      from (assume h₄₁, ad₆ $ ad₃ h₄₁),
    have h₅ : ((ad a g h) or (ad a b c)) → ad (ad i b c) g h,
      from (δ_or₁ h₃ h₄),
    (T₁ h₂ h₅) h₁
```

- ad_4^{ad}

```
theorem ad₄_ad {a b c d e f g : Prop} (h₁ : ad (ad a b c) f g) :
  ad (ad (ad d a (ad d a (ad e a d))) b c) f g :=
  let h := ad d a (ad d a (ad e a d)) in
    have h₂ : (ad (ad a b c) f g) → ((ad a f g) or (ad a b c)),
      from ad₁₀,
    have h₃ : ad a f g → ad (ad h b c) f g,
      from (assume h₃₁, ad₁₁ $ ad₆' $ ad₄ h₃₁),
    have h₄ : ad a b c → ad (ad h b c) f g,
      from (assume h₄₁, ad₆ $ ad₄ h₄₁),
    have h₅ : ((ad a f g) or (ad a b c)) → ad (ad h b c) f g,
      from δ_or₁ h₃ h₄,
    (T₁ h₂ h₅) h₁
```

- ad_5^{ad}

```
theorem ad₅_ad {a b c d e : Prop} (h₁ : ad (ad a b c) d e) : ad (b or a) d e :=
  have h₂ : ad (ad a b c) d e → ((ad a d e) or (ad a b c)),
    from ad₁₀,
  have h₃ : (ad a d e) → (ad (b or a) d e),
    from (assume h, ad₈ $ ad₁₃ $ ad₆' h),
  have h₄ : (ad a b c) → (ad (b or a) d e),
    from (assume h, ad₆ $ ad₅ h),
  have h₅ : ((ad a d e) or (ad a b c)) → (ad (b or a) d e),
    from δ_or₁ h₃ h₄,
  (T₁ h₂ h₅) h₁
```

- ad_6^{ad}

```
theorem ad₆_ad {a b c d e : Prop} (h₁ : ad a d e) : ad (ad a b c) d e :=
  ad₁₁ $ ad₆' h₁
```

- ad_7^{ad}

```
theorem ad₇_ad {a b c d e f : Prop} (h₁ : ad (ad (c or d) a b) e f) :
  ad ((ad c a b) or (ad d a b)) e f :=
  have h₂ : ad (ad (c or d) a b) e f → ((ad (c or d) e f) or (ad (c or d) a b)),
    from ad₁₀,
```

```
  have h₃ : ad (c or d) e f → ((ad c e f) or (ad d e f)),
    from ad₇,
  have h₄ : ad c e f → ad ((ad c a b) or (ad d a b)) e f,
    from (assume h, ad₈ $ ad₆' $ ad₁₁ $ ad₆' h),
  have h₅ : ad d e f → ad ((ad c a b) or (ad d a b)) e f,
    from (assume h, ad₈ $ ad₁₃ $ ad₆' $ ad₁₁ $ ad₆' h),
  have h₆ : ((ad c e f) or (ad d e f)) → ad ((ad c a b) or (ad d a b)) e f,
    from δ_or₁ h₄ h₅,
  have h₇ : ad (c or d) e f → ad ((ad c a b) or (ad d a b)) e f,
    from T₁ h₃ h₆,
  have h₈ : ad (c or d) a b → ad ((ad c a b) or (ad d a b)) e f,
    from (assume h, ad₆ $ ad₇ h),
  have h₉ : ((ad (c or d) e f) or (ad (c or d) a b)) → ad ((ad c a b) or (ad d a b)) e f,
    from δ_or₁ h₇ h₈,
  (T₁ h₂ h₉) h₁
```

- ad_{12}^{ad}

```
theorem ad₁₂_ad {a b c: Prop} (h₁ : ad (a or a) b c) : ad a b c :=
  ad₁₂ $ ad₇ h₁
```

- ad_{13}^{ad}

```
theorem ad₁₃_ad {a b c d : Prop} (h₁ : ad (a or b) c d) : ad (b or a) c d :=
  ad₈ $ ad₁₃ $ ad₇ h₁
```

- ad_{14}^{ad}

```
theorem ad₁₄_ad {a b c d e : Prop} (h₁ : ad (a or (b or c)) d e) : ad ((a or b) or c) d e :=
  ad₈ $ ad₁₃ $ ad₂₂ $ ad₁₃ $ ad₁₄ $ ad₂₁ $ ad₇ h₁
```

- ad_8'

```
theorem ad₈' {a b c d e f : Prop} (h₁ : ad (ad c a b) e f) : ad (ad (c or d) a b) e f :=
  have h₂ : (ad (ad c a b) e f) → ((ad c e f) or (ad c a b)), from ad₁₀,
  have h₃ : (ad c e f) → (ad (ad (c or d) a b) e f), from (assume h, ad₂₇ $ ad₆ $ ad₆_ad h),
  have h₄ : (ad c a b) → (ad (ad (c or d) a b) e f), from (assume h, ad₆ $ ad₆_ad h),
  have h₅ : ((ad c e f) or (ad c a b)) → (ad (ad (c or d) a b) e f), from δ_or₁ h₃ h₄,
  show ad (ad (c or d) a b) e f, from (T₁ h₂ h₅) h₁
```

- ad_8''

```
theorem ad8'' {a b c d e f : Prop} (h1 : ad (ad (c or d) a b) e f) : ad (ad (d or c) a b) e f :=
  have h31 : (ad (ad (c or d) a b) e f) → ((ad (c or d) e f) or (ad (c or d) a b)),
    from ad10,
  have h32 : ad (c or d) e f → ad (ad (d or c) a b) e f,
    from (assume h, ad27 $ ad6 $ ad13_ad h),
  have h33 : ad (c or d) a b → ad (ad (d or c) a b) e f,
    from (assume h, ad6 $ ad13_ad h),
  have h34 : ((ad (c or d) e f) or (ad (c or d) a b)) → ad (ad (d or c) a b) e f,
    from δ_or1 h32 h33,
  (T1 h31 h34) h1
```

- ad_8^{ad}

```
theorem ad8_ad {a b c d e f : Prop} (h1 : ad ((ad c a b) or (ad d a b)) e f) :
  ad (ad (c or d) a b) e f :=
  have h2 : ad ((ad c a b) or (ad d a b)) e f → ((ad (ad c a b) e f) or (ad (ad d a b) e f)),
    from ad7,
  have h3 : ad (ad c a b) e f → ad (ad (c or d) a b) e f,
    from ad8',
  have h4 : ad (ad d a b) e f → ad (ad (c or d) a b) e f,
    from (assume h, ad8'' $ ad8' h),
  have h5 : ((ad (ad c a b) e f) or (ad (ad d a b) e f)) → ad (ad (c or d) a b) e f,
    from δ_or1 h3 h4,
  (T1 h2 h5) h1
```

- ad_9^{ad}

```
theorem ad9_ad {a b c d e f g : Prop} (h1 : ad (ad a b c) f g) (h2 : ad (ad d e a) f g) :
  ad (ad d b c) f g :=
  let g' := ad d f g, c' := ad d b c in
    have h3 : ad (ad a b c) f g → ((ad a f g) or (ad a b c)),
      from ad10,
    have h4 : ad (ad d e a) f g → (g' or (ad d e a)),
      from ad10,
    have h5 : ad a b c → ad d f g → (g' or c'),
      from M1 ad6',
    have h6 : ad a b c → ad d e a → (g' or c'),
      from (assume h, assume i, ad13 $ ad6' $ ad9 h i),
    have h7 : ad a f g → ad d f g → (g' or c'),
      from M1 ad6',
```

```
    have h8 : ad a f g → ad d e a → (g' or c'),
      from (assume h, assume i, ad6' $ ad9 h i),
    have h9 : (g' or (ad d e a)) → ad a b c → (g' or c'),
      from flip (δ_or2 h5 h6),
    have h10 : (g' or (ad d e a)) → ad a f g  → (g' or c'),
      from flip (δ_or2 h7 h8),
    have h11 : (g' or (ad d e a)) → ((ad a f g) or (ad a b c)) → (g' or c'),
      from δ_or2 h10 h9,
    have h12 : (ad (ad a b c) f g) → (g' or (ad d e a)) → (g' or c'),
      from flip (T2 (M1 h3) h11),
    have h13 : (ad (ad a b c) f g) → (ad (ad d e a) f g) → (g' or c'),
      from T2 (M1 h4) h12,
    ad11 (h13 h1 h2)
```

- $\mathrm{ad}_{10}^{\mathrm{ad}}$

```
theorem ad10_ad {a b c d e f g : Prop} (h1 : ad (ad (ad e d c) a b) f g)
  : ad ((ad e a b) or (ad e d c)) f g :=
  have h2 : ad (ad (ad e d c) a b) f g → ((ad (ad e d c) f g) or (ad (ad e d c) a b)),
    from ad10,
  have h3 : ad (ad e d c) f g → ad ((ad e a b) or (ad e d c)) f g,
    from (assume h, ad8 $ ad13 $ ad6' h),
  have h4 : ad (ad e d c) a b → ad ((ad e a b) or (ad e d c)) f g,
    from (assume h, ad6 $ ad10 h),
  have h5 : ((ad (ad e d c) f g) or (ad (ad e d c) a b)) → ad ((ad e a b) or (ad e d c)) f g,
    from δ_or1 h3 h4,
  (T1 h2 h5) h1
```

- $\mathrm{ad}_{11}^{\mathrm{ad}}$

```
theorem ad11_ad {a b c d e f g : Prop} (h1 : ad ((ad e a b) or (ad e d c)) f g)
  : ad (ad (ad e d c) a b) f g :=
  have h2 : ad ((ad e a b) or (ad e d c)) f g → ((ad (ad e a b) f g) or (ad (ad e d c) f g)),
    from ad7,
  have h3 : ad (ad e a b) f g → ((ad e f g) or (ad e a b)),
    from ad10,
  have h4 : ad e f g → ((ad e f g) or (ad e d c)),
    from ad6',
  have h5 : ((ad e f g) or (ad e d c)) → ad (ad (ad e d c) a b) f g,
    from (assume h, ad11 $ ad13 $ ad10 $ ad6 $ ad11 h),
  have h6 : ad e f g → ad (ad (ad e d c) a b) f g,
    from T1 h4 h5,
```

```
have h₇ : ((ad e a b) or (ad e d c)) → ad (ad (ad e d c) a b) f g,
  from (assume h, ad₆ $ ad₁₁ h),
have h₈ : ad e a b → ((ad e a b) or (ad e d c)),
  from ad₆',
have h₉ : ad e a b → ad (ad (ad e d c) a b) f g,
  from T₁ h₈ h₇,
have h₁₀ : ((ad e f g) or (ad e a b)) → ad (ad (ad e d c) a b) f g,
  from δ_or₁ h₆ h₉,
have h₁₁ : ad (ad e a b) f g → ad (ad (ad e d c) a b) f g,
  from T₁ h₃ h₁₀,
have h₁₂ : ad (ad e d c) f g → ad (ad (ad e d c) a b) f g,
  from (assume h, ad₂₇ $ ad₆ $ ad₁₁ $ ad₁₀ h),
have h₁₃ : ((ad (ad e a b) f g) or (ad (ad e d c) f g)) → ad (ad (ad e d c) a b) f g,
  from δ_or₁ h₁₁ h₁₂,
(T₁ h₂ h₁₃) h₁
```

□

Lemma 4.7.4. *Where $\vee := \vee^{\mathsf{ad}}$, the following property holds:*

$$\text{if } A \vee B \in \Gamma^+ \text{ then } A \in \Gamma^+ \text{ or } B \in \Gamma^+$$

Proof. By Lemma 4.7.2, properties $m_\vee$ and $\delta_\vee$ hold in the present calculus, and they, together with the rules of $\mathscr{B}_\vee$, represented in $\mathscr{B}_{\mathsf{ad}}$ by ad_6', ad_{12}, ad_{13} and ad_{14}, imply the desired property (see the proof of Theorem 4.5.6). □

Lemma 4.7.5. *Let $\vee := \vee^{\mathsf{ad}}$. If the rule*

$$\frac{\mathsf{ad}(A_1, D, E) \quad \dots \quad \mathsf{ad}(A_n, D, E)}{\mathsf{ad}(B, D, E)}\ \mathsf{r}^{\mathsf{ad}}$$

is derivable in $\mathscr{B}_{\mathsf{ad}}$, then the rule $\mathsf{r}^{\vee,\mathsf{ad}}$, given by

$$\frac{\mathsf{ad}(C \vee A_1, D, E) \quad \dots \quad \mathsf{ad}(C \vee A_n, D, E)}{\mathsf{ad}(C \vee B, D, E)}\ \mathsf{r}^{\vee,\mathsf{ad}}$$

is also derivable.

Proof. Let r be non-nullary rule given schematically by $(\mathsf{r})\ A_1, \dots, A_n/B$ and define $R := \mathsf{ad}(C, D, E)$. Suppose that r^{ad} holds in $\mathscr{B}_{\mathsf{ad}}$, given by $(\mathsf{r}^{\mathsf{ad}})\ P_1, \dots, P_n/\mathsf{ad}(B, D, E)$, where $P_i = \mathsf{ad}(A_i, D, E)$, thus $\Pi := \{P_i \mid 1 \le i \le n\} \vdash_{\mathscr{B}_{\mathsf{ad}}} \mathsf{ad}(B, D, E)$. By Lemma 4.7.2 and

Lemma 4.5.4, we have (a): $\Pi^{\vee} \vdash_{\mathscr{B}_{\mathsf{ad}}} \mathrm{R} \vee \mathsf{ad}(\mathrm{B},\mathrm{D},\mathrm{E})$, where $\Pi^{\vee} := \{\mathrm{R} \vee \mathrm{P} \mid \mathrm{P} \in \Pi\}$. Let $\mathrm{Q}_i = \mathsf{ad}(\mathrm{C} \vee \mathrm{A}_i, \mathrm{D}, \mathrm{E})$ and $\Theta := \{\mathrm{Q}_i \mid 1 \leq i \leq n\}$, which represent precisely the premisses of the rule whose derivability we want to prove. By rule ad_7, we have (b): $\Theta \vdash_{\mathscr{B}_{\mathsf{ad}}} \mathrm{S}$, for each $\mathrm{S} \in \Pi^{\vee}$. Hence, from (a) and (b), by (T), $\Theta \vdash_{\mathscr{B}_{\mathsf{ad}}} \mathrm{R} \vee \mathsf{ad}(\mathrm{B},\mathrm{D},\mathrm{E})$, which, by rule ad_8 and (T), gives $\Theta \vdash_{\mathscr{B}_{\mathsf{ad}}} \mathsf{ad}(\mathrm{C} \vee \mathrm{B}, \mathrm{D}, \mathrm{E})$, the desired result. □

Corollary 4.7.5.1. *The rules $\mathsf{ad}_{15}^{\mathsf{ad}}, \ldots, \mathsf{ad}_{25}^{\mathsf{ad}}$ are derivable in $\mathscr{B}_{\mathsf{ad}}$.*

Proof. Because the rules $\mathsf{ad}_{15}, \ldots, \mathsf{ad}_{25}$ are respectively the rules $\mathsf{ad}_1^{\vee}, \ldots, \mathsf{ad}_{11}^{\vee}$, and rules $\mathsf{ad}_{\mathsf{j}}^{\mathsf{ad}}$, for $1 \leq \mathsf{j} \leq 11$, were seen to be derivable in Lemma 4.7.3, we note that Lemma 4.7.5 implies that the rules $(\mathsf{ad}_{\mathsf{j}}^{\vee})^{\mathsf{ad}}$ are also derivable. □

Lemma 4.7.6. *The following property holds for $\vdash_{\mathscr{B}_{\mathsf{ad}}}$:*

$$\text{for all } \Gamma \cup \{\mathrm{A},\mathrm{B},\mathrm{C},\mathrm{D}\} \subseteq L_{\mathsf{ad}}.\ \text{if } \Gamma,\mathrm{B},\mathrm{C} \vdash_{\mathscr{B}_{\mathsf{ad}}} \mathrm{A} \text{ then } \Gamma,\mathrm{B} \vdash_{\mathscr{B}_{\mathsf{ad}}} \mathsf{ad}(\mathrm{A},\mathrm{B},\mathrm{C}) \qquad (\delta_{\mathsf{ad}})$$

Proof. Let $\Gamma \cup \{\mathrm{A},\mathrm{B},\mathrm{C}\} \subseteq L_{\mathsf{ad}}$ and suppose that $\Gamma,\mathrm{B},\mathrm{C} \vdash_{\mathscr{B}_{\mathsf{ad}}} \mathrm{A}$, witnessed by a deduction $\mathrm{A}_1, \ldots, \mathrm{A}_n = \mathrm{A}$. Consider the property $P(j)$ meaning the consecution $\Gamma,\mathrm{B} \vdash_{\mathscr{B}_{\mathsf{ad}}} \mathsf{ad}(\mathrm{A}_j,\mathrm{B},\mathrm{C})$. We will prove by induction on this derivation that $P(j)$ holds for all $1 \leq j \leq n$, then, in particular, it will hold for n, which is precisely the desired result. For the base case, there are three possibilities: (a) $\mathrm{A}_1 \in \Gamma$, (b) $\mathrm{A}_1 = \mathrm{B}$ or (c) $\mathrm{A}_1 = \mathrm{C}$. In case of (a), $\Gamma,\mathrm{B} \vdash_{\mathscr{B}_{\mathsf{ad}}} \mathrm{A}_1$, and, by rule ad_6, $\Gamma,\mathrm{B} \vdash_{\mathscr{B}_{\mathsf{ad}}} \mathsf{ad}(\mathrm{A}_1,\mathrm{B},\mathrm{C})$. In case of (b), $\Gamma,\mathrm{B} \vdash_{\mathscr{B}_{\mathsf{ad}}} \mathrm{B}$, by (R) and (M), hence, by rule ad_6, $\Gamma,\mathrm{B} \vdash_{\mathscr{B}_{\mathsf{ad}}} \mathsf{ad}(\mathrm{B},\mathrm{B},\mathrm{C})$. Now, to complete the base case, in case of (c), use $\Gamma,\mathrm{B} \vdash_{\mathscr{B}_{\mathsf{ad}}} \mathrm{B}$ and rule ad_6 to get $\Gamma,\mathrm{B} \vdash_{\mathscr{B}_{\mathsf{ad}}} \mathsf{ad}(\mathrm{B},\mathrm{C},\mathrm{B})$, then rule ad_{13} to get $\Gamma,\mathrm{B} \vdash_{\mathscr{B}_{\mathsf{ad}}} \mathsf{ad}(\mathrm{C},\mathrm{B},\mathrm{C})$, the desired result since $\mathrm{A}_1 = \mathrm{C}$.

For the inductive step, suppose that $P(i)$ holds for all $1 \leq i < k$, where $k > 1$. Then, the same cases considered in the base case apply for A_k and are proved in the same way, together with the case in which A_k results from the application of an instance $\langle \mathrm{A}_{k_1}, \ldots, \mathrm{A}_{k_m}, \mathrm{A}_k \rangle$, with $1 \leq k_l < k$ and $1 \leq l \leq m$, of some of the primitive rules of $\mathscr{B}_{\mathsf{ad}}$, say ad_{s}, some $1 \leq \mathsf{s} \leq 25$. By the induction hypothesis, the following assertions hold: $\Gamma,\mathrm{B} \vdash_{\mathscr{B}_{\mathsf{ad}}} \mathsf{ad}(\mathrm{A}_{k_1},\mathrm{B},\mathrm{C})$, …, $\Gamma,\mathrm{B} \vdash_{\mathscr{B}_{\mathsf{ad}}} \mathsf{ad}(\mathrm{A}_{k_m},\mathrm{B},\mathrm{C})$. Since the lifted version $\mathsf{ad}_{\mathsf{s}}^{\mathsf{ad}}$ is derivable, by Lemma 4.7.3 and Corollary 4.7.5.1, an application of it to $\mathsf{ad}(\mathrm{A}_{k_1},\mathrm{B},\mathrm{C})$,…, $\mathsf{ad}(\mathrm{A}_{k_m},\mathrm{B},\mathrm{C})$ allows us to derive $\mathsf{ad}(\mathrm{A}_k,\mathrm{B},\mathrm{C})$ from $\Gamma \cup \{\mathrm{B}\}$. □

Lemma 4.7.7. *Every set Γ^+ that is Z-maximal with respect to $\vdash_{\mathscr{B}_{\mathsf{ad}}}$ is maximal (consistent).*

Proof. Suppose that Γ^+ is Z-maximal and assume that $\mathrm{A} \notin \Gamma^+$. The goal is to prove that $\Gamma,\mathrm{A} \vdash_{\mathscr{B}_{\mathsf{ad}}} \mathrm{B}$ for every $\mathrm{B} \in L_{\mathsf{ad}}$. Let $\mathrm{C} \in \Gamma^+$ (Lemma 2.6.2 guarantees that Γ^+ is nonempty). An important fact in this proof is that $\Gamma^+ \vdash_{\mathscr{B}_{\mathsf{ad}}} \mathsf{ad}(\mathrm{B},\mathrm{C},\mathrm{Z})$. To prove it, assume that $\Gamma^+ \nvdash_{\mathscr{B}_{\mathsf{ad}}} \mathsf{ad}(\mathrm{B},\mathrm{C},\mathrm{Z})$, in order to derive the contradiction $\Gamma^+ \vdash_{\mathscr{B}_{\mathsf{ad}}} \mathrm{Z}$ by the following reasoning:

(1)	$\Gamma^+, \mathsf{ad}(\mathrm{B},\mathrm{C},\mathrm{Z}) \vdash_{\mathscr{B}_{\mathsf{ad}}} \mathrm{Z}$	Lemma 2.6.1.1
(2)	$\Gamma^+, \mathrm{C}, \mathsf{ad}(\mathrm{B},\mathrm{C},\mathrm{Z}) \vdash_{\mathscr{B}_{\mathsf{ad}}} \mathrm{Z}$	1 (M)
(3)	$\Gamma^+, \mathrm{C} \vdash_{\mathscr{B}_{\mathsf{ad}}} \mathsf{ad}(\mathrm{Z},\mathrm{C},\mathsf{ad}(\mathrm{B},\mathrm{C},\mathrm{Z}))$	δ_{ad}
(4)	$\mathrm{C} \vdash_{\mathscr{B}_{\mathsf{ad}}} \mathrm{C}$	(R)
(5)	$\Gamma^+, \mathrm{C} \vdash_{\mathscr{B}_{\mathsf{ad}}} \mathrm{C}$	4 (M)
(6)	$\Gamma^+, \mathrm{C} \vdash_{\mathscr{B}_{\mathsf{ad}}} \mathsf{ad}(\mathrm{Z},\mathrm{C},\mathsf{ad}(\mathrm{Z},\mathrm{C},\mathsf{ad}(\mathrm{B},\mathrm{C},\mathrm{Z})))$	5 ad'_4
(7)	$\Gamma^+, \mathrm{C} \vdash_{\mathscr{B}_{\mathsf{ad}}} \mathrm{Z}$	3, 6 ad_1
(8)	$\Gamma^+ \vdash_{\mathscr{B}_{\mathsf{ad}}} \mathrm{Z}$	7 $\mathrm{C} \in \Gamma^+$

Because $\Gamma^+ \vdash_{\mathscr{B}_{\mathsf{ad}}} \mathsf{ad}(\mathrm{B},\mathrm{C},\mathrm{Z})$, we get (a): $\Gamma^+, \mathrm{A} \vdash_{\mathscr{B}_{\mathsf{ad}}} \mathsf{ad}(\mathrm{B},\mathrm{C},\mathrm{Z})$ by (M). Since $\mathrm{A} \notin \Gamma^+$, we also have (b): $\Gamma^+, \mathrm{A} \vdash_{\mathscr{B}_{\mathsf{ad}}} \mathrm{Z}$. Then these consecutions, by ad_1, yield $\Gamma^+, \mathrm{A} \vdash_{\mathscr{B}_{\mathsf{ad}}} \mathrm{B}$, proving that Γ^+ is maximal. □

Theorem 4.7.8. *The calculus $\mathscr{B}_{\mathsf{ad}}$ is complete with respect to the matrix $\mathbb{2}_{\mathsf{ad}}$.*

Proof. Let $\Gamma \cup \{\mathrm{Z}\} \subseteq L_{\mathsf{ad}}$ such that $\Gamma \nvdash_{\mathscr{B}_{\mathsf{ad}}} \mathrm{Z}$ and take a Z-maximal theory $\Gamma^+ \supseteq \Gamma$ by Lindenbaum-Asser Lemma. From the truth-table of ad and the formulation given in Section 2.7, the completeness property (ad) is given by:

$$\mathsf{ad}(\mathrm{A},\mathrm{B},\mathrm{C}) \in \Gamma^+ \text{ iff } \mathrm{A} \in \Gamma^+ \text{ or } (\mathrm{B} \in \Gamma^+ \text{ and } \mathrm{C} \notin \Gamma^+) \qquad (\mathsf{ad})$$

In the left to right direction, suppose that $\mathsf{ad}(\mathrm{A},\mathrm{B},\mathrm{C}) \in \Gamma^+$, thus (a): $\Gamma^+ \vdash_{\mathscr{B}_{\mathsf{ad}}} \mathsf{ad}(\mathrm{A},\mathrm{B},\mathrm{C})$. Then, by rule ad_5 and (T), $\Gamma^+ \vdash_{\mathscr{B}_{\mathsf{ad}}} \mathsf{ad}(\mathrm{B},\mathrm{A},\mathrm{B})$, which, by Lemma 4.7.4, implies $\Gamma^+ \vdash_{\mathscr{B}_{\mathsf{ad}}} \mathrm{A}$ or $\Gamma^+ \vdash_{\mathscr{B}_{\mathsf{ad}}} \mathrm{B}$. We proceed by considering the cases regarding the derivability of A from Γ^+. In case $\Gamma^+ \vdash_{\mathscr{B}_{\mathsf{ad}}} \mathrm{A}$, there is nothing to be done. Otherwise, if $\Gamma^+ \nvdash_{\mathscr{B}_{\mathsf{ad}}} \mathrm{A}$, we have $\Gamma^+ \vdash_{\mathscr{B}_{\mathsf{ad}}} \mathrm{B}$ because of the lemma just mentioned. For the sake of contradiction, suppose that $\mathrm{C} \in \Gamma^+$, thus (b): $\Gamma^+ \vdash_{\mathscr{B}_{\mathsf{ad}}} \mathrm{C}$. From (a) and (b), by ad_1, we get $\Gamma^+ \vdash_{\mathscr{B}_{\mathsf{ad}}} \mathrm{A}$, contradicting the assumption that $\Gamma^+ \nvdash_{\mathscr{B}_{\mathsf{ad}}} \mathrm{A}$. Therefore $\mathrm{C} \notin \Gamma^+$, as desired.

From right-to-left, in case $\mathrm{A} \in \Gamma^+$, $\Gamma^+ \vdash_{\mathscr{B}_{\mathsf{ad}}} \mathrm{A}$. From this, by rule ad_6 and (T), we have $\Gamma^+ \vdash_{\mathscr{B}_{\mathsf{ad}}} \mathsf{ad}(\mathrm{A},\mathrm{B},\mathrm{C})$. In case $\mathrm{B} \in \Gamma^+$ and $\mathrm{C} \notin \Gamma^+$, because Γ^+ is also maximal (Lemma 4.7.7), $\Gamma^+, \mathrm{B}, \mathrm{C} \vdash_{\mathscr{B}_{\mathsf{ad}}} \mathrm{A}$, which, by the deduction theorem δ_{ad}, leads to $\Gamma^+, \mathrm{B} \vdash_{\mathscr{B}_{\mathsf{ad}}} \mathsf{ad}(\mathrm{A},\mathrm{B},\mathrm{C})$ and thus to $\Gamma^+ \vdash_{\mathscr{B}_{\mathsf{ad}}} \mathsf{ad}(\mathrm{A},\mathrm{B},\mathrm{C})$, since $\mathrm{B} \in \Gamma^+$ by assumption. □

The expansion $\mathcal{B}_{\mathsf{ad},\top}$ axiomatized by the calculus below, in view of Corollary 2.8.4.1:

Hilbert Calculus 21. $\mathscr{B}_{\mathsf{ad},\top}$

$$\mathscr{B}_{\mathsf{ad}} \quad \mathscr{B}_{\top}$$

4.8 $\mathcal{B}_{\neg}$, $\mathcal{B}_{\neg,\top}$

The axiomatization here presented for $\mathcal{B}_{\neg}$, the fragment of pure classical negation, uses the rule of explosion, denoted by n_1, and the rules for double negation introduction and elimination:

Hilbert Calculus 22. $\mathscr{B}_{\neg}$

$$\frac{\mathrm{A} \quad \neg\mathrm{A}}{\mathrm{B}}\ \mathsf{n}_1 \qquad \frac{\mathrm{A}}{\neg\neg\mathrm{A}}\ \mathsf{n}_2 \qquad \frac{\neg\neg\mathrm{A}}{\mathrm{A}}\ \mathsf{n}_3$$

Theorem 4.8.1. *The calculus $\mathscr{B}_{\neg}$ is sound with respect to the matrix $\mathbb{2}_{\neg}$.*

Proof. Let v be an arbitrary $\mathbb{2}_{\neg}$-valuation. By the interpretation of $\neg$ in $\mathbb{2}_{\neg}$, the premisses of n_1 can not be simultaneously evaluated to 1. Soundness of rules n_2 and n_3 follows by the involutive characteristic of $\neg^{\mathbb{2}_{\neg}}$, that is, the fact that $\neg^{\mathbb{2}_{\neg}}(\neg^{\mathbb{2}_{\neg}}(x)) = x$. □

We now intend to prove a deduction theorem for negation, which will have as consequence the completeness result for the calculus $\mathscr{B}_{\neg}$ with respect to the pure negation fragment. For that, we first prove a lemma regarding the structure of the formulas in a derivation in the calculus under discussion. This result will then be used to prove the desired theorem.

Lemma 4.8.2. *Let $\mathrm{B}_1, \ldots, \mathrm{B}_n = \mathrm{B}$ be a derivation of B from Γ. If the rule n_1 was not applied to obtain any of the formulas $\mathrm{B}_1, \ldots, \mathrm{B}_k$, with $1 \leq k \leq n$, then each B_j, for $1 \leq j \leq k$, has the form $\neg^{2n+c}\mathrm{C}$, where $\neg^{2m+c}\mathrm{C} \in \Gamma$, $c \in \{0,1\}$, C is not $\neg$-headed, and $m, n \in \omega$.*

Proof. Let $1 \leq k \leq n$, and suppose that the rule n_1 was not applied to derive any of the formulas B_j in the derivation of B from Γ, where $1 \leq j \leq k$. The proof goes by induction on

j, considering $P(j)$ given by the statement "B_j has the form $\neg^{2n+c}\mathrm{C}$, for some $\neg^{2m+c}\mathrm{C} \in \Gamma$, where $c \in \{0,1\}$, C not $\neg$-headed and $m, n \in \omega$". The base case, when $j = 1$, presupposes checking only the case when $\mathrm{B}_1 \in \Gamma$, since this calculus contains no axioms. Then, if $\mathrm{B}_1 \in \Gamma$, the property follows because any formula in a language whose signature has $\neg$ as unary symbol has the form $\neg^{2m+c}\mathrm{C}$, for some non-$\neg$-headed formula C, $c \in \{0,1\}$ and $m \in \omega$. Now, suppose that the claim holds for all B_i, with $i < j$. In case $\mathrm{B}_j \in \Gamma$, the same argument used in the base case is applicable. Otherwise, B_j follows either from n_2 or n_3 applied to B_{i_1}, with $i_1 < j$. By the induction hypothesis, $\mathrm{B}_{i_1} = \neg^{2n_1+c_1}\mathrm{C} \in \Gamma$ for some non-$\neg$-headed formula C, $n_1 \in \omega$ and $c_1 \in \{0,1\}$. In the first case, B_j has the form $\neg\neg\mathrm{B}_{i_1}$, as a result of the application of rule n_2, thus $\mathrm{B}_j = \neg^{2n_1+2+c_1}\mathrm{C} = \neg^{2(n_1+1)+c_1}\mathrm{C} = \neg^{2n_1'+c_1}\mathrm{C}$, with $n_1' \in \omega$. In the second case, B_j is a formula D, as a result of rule n_3 applied to $\mathrm{B}_{i_1} = \neg\neg\mathrm{D}$. This forces n_1 to be strictly positive, since $c_1 \in \{0,1\}$. This fact implies that $\mathrm{B}_j = \neg^{2n_1+c_1-2}\mathrm{C} = \neg^{2(n_1-1)+c_1}\mathrm{C} = \neg^{2n_1'+c_1}\mathrm{C}$, where $n_1' \in \omega$ necessarily. □

Theorem 4.8.3. *The following property holds for* $\vdash_{\mathscr{B}_\neg}$*:*

$$\text{for all } \Gamma \cup \{\mathrm{B}\} \subseteq L_\neg. \text{if } \Gamma, \neg\mathrm{B} \vdash_{\mathscr{B}_\neg} \mathrm{B} \text{ then } \Gamma \vdash_{\mathscr{B}_\neg} \mathrm{B} \qquad (\delta_\neg)$$

Proof. Let $\Gamma \cup \{\mathrm{B}\} \subseteq L_\neg$. Suppose that $\Gamma, \neg\mathrm{B} \vdash_{\mathscr{B}_\neg} \mathrm{B}$ and that this is witnessed by the deduction $\mathrm{B}_1, \ldots, \mathrm{B}_n = \mathrm{B}$. Notice that, if the rule n_1 was not applied in this entire derivation, then Lemma 4.8.2 guarantees that $\mathrm{B} = \neg^{2n+c}\mathrm{C}$, where $\mathrm{B}' = \neg^{2m+c}\mathrm{C} \in \Gamma$, C is not $\neg$-headed, $m, n \in \omega$ and $c \in \{0,1\}$. In this case, there are three cases to consider: if $m = n$, then $\mathrm{B} = \mathrm{B}' \in \Gamma$, thus $\Gamma \vdash_{\mathscr{B}_\neg} \mathrm{B}$; if $m < n$, then perform consecutive $n - m$ applications of rule n_2 starting from B', deriving B; finally, if $m > n$, do the same as the latter case, but with $m - n$ applications of rule n_3.

Now, suppose that the rule n_1 was applied for the first time in step k, where $k \geq 3$, resulting in the formula B_k. In this case, there are formulas B_{k_1} and B_{k_2}, where $k_1, k_2 < k$, such that (a): $\mathrm{B}_{k_2} = \neg\mathrm{B}_{k_1}$. Because of Lemma 4.8.2, (b): $\mathrm{B}_{k_1} = \neg^{2n_1+c_1}\mathrm{C}_1$ and (c): $\mathrm{B}_{k_2} = \neg^{2n_2+c_2}\mathrm{C}_2$, where (d): $\neg^{2m_1+c_1}\mathrm{C}_1 \in \Gamma$ and (e): $\neg^{2m_2+c_2}\mathrm{C}_2 \in \Gamma$, C_1 and C_2 are not $\neg$-headed, $c_1, c_2 \in \{0,1\}$ and $n_1, m_1, n_2, m_2 \in \omega$. The facts (a), (b) and (c) force $\mathrm{C}_1 = \mathrm{C}_2$ and establish that $2n_2 + c_2 = 2n_1 + c_1 + 1$. The latter implies $c_1 \neq c_2$, otherwise $2n_2 = 2n_1 + 1$, an absurd. Without loss of generality, take $c_1 = 0$ and $c_2 = 1$, then, from (d) and (e), $\neg^{2m_1}\mathrm{C}_1 \in \Gamma$ and $\neg^{2m_2+1}\mathrm{C}_2 = \neg^{2m_2+1}\mathrm{C}_1 \in \Gamma$. Hence, by m_1 consecutive applications of rule n_3 starting from $\neg^{2m_1}\mathrm{C}_1$, we derive C_1 from Γ. Similarly, with m_2 consecutive applications of this same rule starting from $\neg^{2m_2+1}\mathrm{C}_1$, we get $\neg\mathrm{C}_1$ from Γ. These two derivations are enough ingredients to produce B from Γ by rule n_1, which is the desired result. □

Theorem 4.8.4. *The calculus $\mathscr{B}_\neg$ is complete with respect to the matrix $\mathbb{2}_\neg$.*

Proof. Let $\Gamma \cup \{Z\} \subseteq L_\neg$. Consider $\Gamma^+ \supseteq \Gamma$ a Z-maximal set and the completeness property for $\neg$ presented below, obtained following the procedure described in Section 2.7:

$$\neg A \in \Gamma^+ \text{ iff } A \notin \Gamma^+. \tag{$\neg$}$$

To prove this property, from the left to the right, we work by contradiction, so suppose that $\neg A \in \Gamma^+$ and $A \in \Gamma^+$. Since Γ^+ is deductively closed (see Corollary 2.6.1.2), $\Gamma^+ \vdash_{\mathscr{B}_\neg} A$ and $\Gamma^+ \vdash_{\mathscr{B}_\neg} \neg A$. Then, by an appeal to n_1, $\Gamma^+ \vdash_{\mathscr{B}_\neg} Z$, contradicting the fact that $\Gamma^+ \nvdash_{\mathscr{B}_\neg} Z$. From the right to the left, again by contradiction, suppose that $A \notin \Gamma^+$ and $\neg A \notin \Gamma^+$. Then, by Theorem 2.6.1, (a): $\Gamma^+, A \vdash_{\mathscr{B}_\neg} Z$ and (b): $\Gamma^+, \neg A \vdash_{\mathscr{B}_\neg} Z$. Finally, the following reasoning produces the absurd $\Gamma^+ \vdash_{\mathscr{B}_\neg} Z$:

(1)	$\Gamma^+, \neg A \vdash_{\mathscr{B}_\neg} Z$	(a)
(2)	$\neg Z, \varphi \vdash_{\mathscr{B}_\neg} A$	n_1
(3)	$\Gamma^+, \neg Z, \neg A \vdash_{\mathscr{B}_\neg} A$	1, 2 (T)
(4)	$\Gamma^+, \neg Z \vdash_{\mathscr{B}_\neg} A$	3 Theorem 4.8.3
(5)	$\Gamma^+, A \vdash_{\mathscr{B}_\neg} Z$	(b)
(6)	$\Gamma^+, \neg Z \vdash_{\mathscr{B}_\neg} \varphi$	4, 5 (T)
(7)	$\Gamma^+ \vdash_{\mathscr{B}_\neg} Z$	6 Theorem 4.8.3

□

We finish this section with the calculus for the expansion $\mathcal{B}_{\neg,\top}$, which is, as usual, axiomatized by adding the rule t_1 (see Corollary 2.8.4.1):

Hilbert Calculus 23. $\mathscr{B}_{\neg,\top}$

$\mathscr{B}_\neg \quad \mathscr{B}_\top$

In order to keep the deduction theorem and, consequently, the completeness property for negation, we just need a slight modification in the statement of Lemma 4.8.2: instead of "$\neg^{2m+c}C \in \Gamma$", we would write "$\Gamma \vdash_{\mathscr{B}_{\neg,\top}} \neg^{2m+c}C$", something that leads to small modifications in the proof of Theorem 4.8.3. The proof of the modified version of the referred lemma is very similar to the proof of the original one.

4.9 $\mathcal{B}_{\mathsf{pt}}, \mathcal{B}_{\mathsf{pt},\bot}, \mathcal{B}_{\mathsf{pt},\top}, \mathcal{B}_{\mathsf{pt},\bot,\top}$

The classical connective pt may be defined from those in $\mathcal{B}$ by means of the translation $\mathbf{t}(\mathsf{pt}) = \lambda p, q, r.p + (q + r)$, where $+ = \lambda p, q.(p \wedge \neg q) \vee (q \wedge \neg p)$. Notice that the induced interpretation $\mathsf{pt}^{\mathbb{2}}$ is such that $\mathsf{pt}^{\mathbb{2}}(x, y, z) = 1$ if and only if either $x = y = z = 1$ or exactly one of the elements x, y and z is 1, what characterizes it as a linear boolean function. The candidate axiomatization for the fragment $\mathcal{B}_{\mathsf{pt}}$ is presented below and clearly reflects the behaviour of $\mathsf{pt}^{\mathbb{2}}$. The proof of soundness is presented right after it.

Hilbert Calculus 24. $\mathscr{B}_{\mathsf{pt}}$

$$\frac{A \quad B \quad C}{\mathsf{pt}(A, B, C)}\ \mathsf{pt}_1 \qquad \frac{\mathsf{pt}(A, B, C)}{\mathsf{pt}(B, A, C)}\ \mathsf{pt}_2 \qquad \frac{\mathsf{pt}(A, B, C)}{\mathsf{pt}(A, C, B)}\ \mathsf{pt}_3$$

$$\frac{A}{\mathsf{pt}(A, B, B)}\ \mathsf{pt}_4 \qquad \frac{\mathsf{pt}(A, B, B)}{A}\ \mathsf{pt}_5 \qquad \frac{\mathsf{pt}(A, B, \mathsf{pt}(C, D, E))}{\mathsf{pt}(\mathsf{pt}(A, B, C), D, E)}\ \mathsf{pt}_6$$

Theorem 4.9.1. *The calculus $\mathscr{B}_{\mathsf{pt}}$ is sound with respect to the matrix $\mathbb{2}_{\mathsf{pt}}$.*

Proof. Let v be an arbitrary $\mathbb{2}_{\mathsf{pt}}$-valuation. For rule pt_1, if $v(A) = 1$, $v(B) = 1$ and $v(C) = 1$, then $v(\mathsf{pt}(A, B, C)) = 1$. For rules pt_2 and pt_3, if $v(\mathsf{pt}(A, B, C)) = 1$, then either all components all assigned the value 1 or only one of them is assigned 1. In any case, permuting the order in which they occur in the compound does not its value under v. For rule pt_4, if $v(A) = 1$, consider two cases: either $v(B) = 1$ or $v(B) = 0$; the former implies that all of subformulas of $\mathsf{pt}(A, B, B)$ are assigned the value 1 and the latter implies that only one subformula is assigned the value 1, so $v(\mathsf{pt}(A, B, B)) = 1$. The argument is analogous to for pt_5. Finally, for rule pt_6, if v assigns 0 to $\mathsf{pt}(\mathsf{pt}(A, B, C), D, E)$, consider the following cases:

- if $v(\mathsf{pt}(A, B, C)) = 0$, $v(D) = 0$ and $v(E) = 0$, we have these subcases:
  - if $v(A) = 0$, $v(B) = 0$ and $v(C) = 0$, then v assigns necessarily 0 to the premiss;
  - if $v(A) = 1$ and $v(B) = 1$ but $v(C) = 0$, then $v(\mathsf{pt}(C, D, E)) = 0$; and
  - if $v(A) = 0$ and $v(B) = 1$ but $v(C) = 1$, or $v(A) = 1$ and $v(B) = 0$ but $v(C) = 1$, then $v(\mathsf{pt}(C, D, E)) = 1$, causing v to assign 0 to the premiss.
- if $v(\mathsf{pt}(A, B, C)) = 1$, $v(D) = 1$ and $v(E) = 0$, we have these subcases:

– if $v(\mathrm{A}) = 1$, $v(\mathrm{B}) = 1$ and $v(\mathrm{C}) = 1$, then the premiss gets the value 0 immediately;

– if $v(\mathrm{A}) = 1$, $v(\mathrm{B}) = 0$ and $v(\mathrm{C}) = 0$, then $v(\mathsf{pt}(\mathrm{C},\mathrm{D},\mathrm{E})) = 1$, causing v to give the value 0 to the premiss;

– if $v(\mathrm{A}) = 0$, $v(\mathrm{B}) = 1$ and $v(\mathrm{C}) = 0$, the argument is analogous to the previous case;

– if $v(\mathrm{A}) = 0$, $v(\mathrm{B}) = 0$ and $v(\mathrm{C}) = 1$, since $v(\mathrm{C}) = 1$ and $v(\mathrm{D}) = 1$, $v(\mathsf{pt}(\mathrm{C},\mathrm{D},\mathrm{E})) = 0$, and the premiss get the value 0.

- The remaining cases go analogously as the previous case-by-case analysis.

□

In what follows, if r is an n-ary rule, with $n \in \omega$, let r^{pt}, the pt-lifted version of r, be the rule given by the set of instances $\langle \mathsf{pt}(\mathrm{C},\mathrm{D},\mathrm{A}_1),\ldots,\mathsf{pt}(\mathrm{C},\mathrm{D},\mathrm{A}_n),\mathsf{pt}(\mathrm{C},\mathrm{D},\mathrm{B})\rangle$, where $\langle \mathrm{A}_1,\ldots,\mathrm{A}_n,\mathrm{B}\rangle$ is an instance of r and $\mathrm{C},\mathrm{D} \in L_{\mathsf{pt}}$. We proceed by deriving some rules in $\mathscr{B}_{\mathsf{pt}}$. Some of them are the pt-lifted versions of the primitive rules, while the others will simplify the proofs of important properties in the path to the completeness result.

Lemma 4.9.2. *The following rules are derivable in* $\mathscr{B}_{\mathsf{pt}}$*:*

$$\frac{\mathsf{pt}(\mathsf{pt}(\mathrm{A},\mathrm{B},\mathrm{C}),\mathrm{D},\mathrm{E})}{\mathsf{pt}(\mathrm{A},\mathrm{B},\mathsf{pt}(\mathrm{C},\mathrm{D},\mathrm{E}))}\ \mathsf{pt}_7$$

$$\frac{\mathsf{pt}(\mathrm{D},\mathrm{E},\mathsf{pt}(\mathrm{A},\mathrm{B},\mathrm{C}))}{\mathsf{pt}(\mathrm{D},\mathrm{E},\mathsf{pt}(\mathrm{B},\mathrm{A},\mathrm{C}))}\ \mathsf{pt}_2^{\mathsf{pt}}$$

$$\frac{\mathsf{pt}(\mathrm{D},\mathrm{E},\mathsf{pt}(\mathrm{A},\mathrm{B},\mathrm{C}))}{\mathsf{pt}(\mathrm{D},\mathrm{E},\mathsf{pt}(\mathrm{A},\mathrm{C},\mathrm{B}))}\ \mathsf{pt}_3^{\mathsf{pt}}$$

$$\frac{\mathsf{pt}(\mathrm{C},\mathrm{D},\mathrm{A})}{\mathsf{pt}(\mathrm{C},\mathrm{D},\mathsf{pt}(\mathrm{A},\mathrm{B},\mathrm{B}))}\ \mathsf{pt}_4^{\mathsf{pt}}$$

$$\frac{\mathsf{pt}(\mathrm{C},\mathrm{D},\mathsf{pt}(\mathrm{A},\mathrm{B},\mathrm{B}))}{\mathsf{pt}(\mathrm{C},\mathrm{D},\mathrm{A})}\ \mathsf{pt}_5^{\mathsf{pt}}$$

$$\frac{\mathsf{pt}(\mathrm{F},\mathrm{G},\mathsf{pt}(\mathrm{A},\mathrm{B},\mathsf{pt}(\mathrm{C},\mathrm{D},\mathrm{E})))}{\mathsf{pt}(\mathrm{F},\mathrm{G},\mathsf{pt}(\mathsf{pt}(\mathrm{A},\mathrm{B},\mathrm{C}),\mathrm{D},\mathrm{E}))}\ \mathsf{pt}_6^{\mathsf{pt}}$$

$$\frac{\mathsf{pt}(\mathsf{pt}(\mathrm{A},\mathrm{B},\mathrm{C}),\mathrm{A},\mathrm{B})}{\mathrm{C}}\ \mathsf{pt}_8$$

$$\frac{\mathsf{pt}(\mathsf{pt}(\mathrm{A},\mathrm{B},\mathrm{C}),\mathrm{A},\mathrm{C})}{\mathrm{B}}\ \mathsf{pt}_9$$

$$\frac{\mathsf{pt}(\mathsf{pt}(\mathrm{A},\mathrm{B},\mathrm{C}),\mathrm{B},\mathrm{C})}{\mathrm{A}}\ \mathsf{pt}_{10}$$

$$\frac{\mathsf{pt}(E, F, \mathsf{pt}(\mathsf{pt}(A, B, C), C, D))}{\mathsf{pt}(E, F, \mathsf{pt}(A, B, D))}\ \mathsf{pt}'_{11}$$

$$\frac{\mathsf{pt}(E, F, \mathsf{pt}(\mathsf{pt}(A, B, C), B, D))}{\mathsf{pt}(E, F, \mathsf{pt}(A, C, D))}\ \mathsf{pt}''_{11}$$

$$\frac{\mathsf{pt}(E, F, \mathsf{pt}(\mathsf{pt}(A, B, C), A, D))}{\mathsf{pt}(E, F, \mathsf{pt}(B, C, D))}\ \mathsf{pt}'''_{11}$$

$$\frac{\mathsf{pt}(\mathsf{pt}(\mathsf{pt}(A, B, C), A, D), \mathsf{pt}(\mathsf{pt}(A, B, C), B, D), \mathsf{pt}(\mathsf{pt}(A, B, C), C, D))}{D}\ \mathsf{pt}_{11}$$

$$\frac{\mathsf{pt}(A, B, C) \quad D \quad E}{\mathsf{pt}(A, B, \mathsf{pt}(C, D, E))}\ \mathsf{pt}_{12}$$

$$\frac{\mathsf{pt}(A, B, C) \quad \mathsf{pt}(A, B, D) \quad E}{\mathsf{pt}(C, D, E)}\ \mathsf{pt}_{13}$$

$$\frac{\mathsf{pt}(A, B, C) \quad \mathsf{pt}(A, B, D) \quad \mathsf{pt}(A, B, E)}{\mathsf{pt}(A, B, \mathsf{pt}(C, D, E))}\ \mathsf{pt}_{14}$$

$$\frac{A}{\mathsf{pt}(\mathsf{pt}(A, B, C), B, C)}\ \mathsf{pt}'_{4}$$

Proof. The formally verified derivation of each rule is presented below, following what was explained in Chapter 3.

- pt_7

```
theorem pt₇ {a b c d e : Prop} (h₁ : pt (pt a b c) d e) : pt a b (pt c d e) :=
  have h₂ : pt d (pt a b c) e, from pt₂ h₁,
  have h₃ : pt d e (pt a b c), from pt₃ h₂,
  have h₄ : pt (pt d e a) b c, from pt₆ h₃,
  have h₅ : pt b (pt d e a) c, from pt₂ h₄,
  have h₆ : pt b c (pt d e a), from pt₃ h₅,
  have h₇ : pt (pt b c d) e a, from pt₆ h₆,
  have h₈ : pt e (pt b c d) a, from pt₂ h₇,
  have h₉ : pt e a (pt b c d), from pt₃ h₈,
  have h₁₀ : pt (pt e a b) c d, from pt₆ h₉,
  have h₁₁ : pt c (pt e a b) d, from pt₂ h₁₀,
  have h₁₂ : pt c d (pt e a b), from pt₃ h₁₁,
  have h₁₃ : pt (pt c d e) a b, from pt₆ h₁₂,
  have h₁₄ : pt a (pt c d e) b, from pt₂ h₁₃,
  show pt a b (pt c d e), from pt₃ h₁₄
```

- $\mathsf{pt}_2^{\mathsf{pt}}$

```
theorem pt₂_pt {a b c d e : Prop} (h₁ : pt d e (pt a b c)) : pt d e (pt b a c) :=
```

```
  have h₂ : pt d (pt a b c) e, from pt₃ h₁,
  have h₃ : pt (pt a b c) d e, from pt₂ h₂,
  have h₄ : pt a b (pt c d e), from pt₇ h₃,
  have h₅ : pt b a (pt c d e), from pt₂ h₄,
  have h₆ : pt (pt b a c) d e, from pt₆ h₅,
  have h₇ : pt d (pt b a c) e, from pt₂ h₆,
  show pt d e (pt b a c), from pt₃ h₇
```

- pt_3^{pt}

```
theorem pt₃_pt {a b c d e : Prop} (h₁ : pt d e (pt a b c)) : pt d e (pt a c b) :=
  have h₂ : pt (pt d e a) b c, from pt₆ h₁,
  have h₃ : pt (pt d e a) c b, from pt₃ h₂,
  show pt d e (pt a c b), from pt₇ h₃
```

- pt_4^{pt}

```
theorem pt₄_pt {a b c d : Prop} (h₁ : pt c d a) : pt c d (pt a b b) :=
  have h₂ : pt (pt c d a) b b, from pt₄ h₁,
  show pt c d (pt a b b), from pt₇ h₂
```

- pt_5^{pt}

```
theorem pt₅_pt {a b c d : Prop} (h₁ : pt c d (pt a b b)) : pt c d a :=
  have h₂ : pt (pt c d a) b b, from pt₆ h₁,
  show pt c d a, from pt₅ h₂
```

- pt_6^{pt}

```
theorem pt₆_pt {a b c d e f g : Prop}
  (h₁ : pt f g (pt a b (pt c d e)))
  : pt f g (pt (pt a b c) d e) :=
  have h₂ : pt (pt a b (pt c d e)) f g, from pt₂ (pt₃ h₁),
  have h₃ : pt a b (pt (pt c d e) f g), from pt₇ h₂,
  have h₄ : pt (pt (pt c d e) f g) a b, from pt₂ (pt₃ h₃),
  have h₅ : pt (pt c d e) f (pt g a b), from pt₇ h₄,
  have h₆ : pt c d (pt e f (pt g a b)), from pt₇ h₅,
  have h₇ : pt (pt e f (pt g a b)) c d, from pt₂ (pt₃ h₆),
  have h₈ : pt e f (pt (pt g a b) c d), from pt₇ h₇,
  have h₉ : pt (pt (pt g a b) c d) e f, from pt₂ (pt₃ h₈),
```

```
    have h₁₀ : pt (pt g a b) c (pt d e f), from pt₇ h₉,
    have h₁₁ : pt g a (pt b c (pt d e f)), from pt₇ h₁₀,
    have h₁₂ : pt (pt b c (pt d e f)) g a, from pt₂ (pt₃ h₁₁),
    have h₁₃ : pt b c (pt (pt d e f) g a), from pt₇ h₁₂,
    have h₁₄ : pt (pt (pt d e f) g a) b c, from pt₂ (pt₃ h₁₃),
    have h₁₅ : pt (pt d e f) g (pt a b c), from pt₇ h₁₄,
    have h₁₆ : pt d e (pt f g (pt a b c)), from pt₇ h₁₅,
    have h₁₇ : pt (pt f g (pt a b c)) d e, from pt₂ (pt₃ h₁₆),
    show pt f g (pt (pt a b c) d e), from pt₇ h₁₇
```

- pt_8

```
theorem pt₈ {a b c : Prop} (h₁ : pt (pt a b c) a b) : c :=
    have h₂ : pt a (pt a b c) b, from pt₂ h₁,
    have h₃ : pt a b (pt a b c), from pt₃ h₂,
    have h₄ : pt a b (pt a c b), from pt₃_pt h₃,
    have h₅ : pt a b (pt c a b), from pt₂_pt h₄,
    have h₆ : pt a (pt c a b) b, from pt₃ h₅,
    have h₇ : pt (pt c a b) a b, from pt₂ h₆,
    have h₈ : pt c a (pt b a b), from pt₇ h₇,
    have h₉ : pt c a (pt a b b), from pt₂_pt h₈,
    have h₁₀ : pt c a a, from pt₅_pt h₉,
    show c, from pt₅ h₁₀
```

- pt_9

```
theorem pt₉ {a b c : Prop} (h₁ : pt (pt a b c) a c) : b :=
    have h₂ : pt a b (pt c a c), from pt₇ h₁,
    have h₃ : pt a b (pt a c c), from pt₂_pt h₂,
    have h₄ : pt a b a, from pt₅_pt h₃,
    have h₅ : pt b a a, from pt₂ h₄,
    show b, from pt₅ h₅
```

- pt_{10}

```
theorem pt₁₀ {a b c : Prop} (h₁ : pt (pt a b c) b c) : a :=
    have h₂ : pt a b (pt c b c), from pt₇ h₁,
    have h₃ : pt a b (pt b c c), from pt₂_pt h₂,
    have h₄ : pt a b b, from pt₅_pt h₃,
    show a, from pt₅ h₄
```

- pt'_{11}

```
lemma pt₁₁' {a b c d e f : Prop} (h₁ : pt e f (pt (pt a b c) c d))
  : pt e f (pt a b d) :=
  have h₂ : pt e f (pt c d (pt a b c)), from pt₃_pt (pt₂_pt h₁),
  have h₃ : pt e f (pt (pt c d a) b c), from pt₆_pt h₂,
  have h₄ : pt e f (pt b c (pt c d a)), from pt₃_pt (pt₂_pt h₃),
  have h₅ : pt e f (pt (pt b c c) d a), from pt₆_pt h₄,
  have h₆ : pt e f (pt d a (pt b c c)), from pt₃_pt (pt₂_pt h₅),
  have h₇ : pt e f (pt (pt d a b) c c), from pt₆_pt h₆,
  have h₈ : pt e f (pt d a b), from pt₅_pt h₇,
  show pt e f (pt a b d), from pt₃_pt (pt₂_pt h₈)
```

- pt''_{11}

```
lemma pt₁₁'' {a b c d e f : Prop} (h₁ : pt e f (pt (pt a b c) b d))
  : pt e f (pt a c d) :=
  have h₂ : pt e f (pt b d (pt a b c)), from pt₃_pt (pt₂_pt h₁),
  have h₃ : pt e f (pt (pt b d a) b c), from pt₆_pt h₂,
  have h₄ : pt e f (pt c b (pt b d a)), from pt₂_pt (pt₃_pt (pt₂_pt h₃)),
  have h₅ : pt e f (pt (pt c b b) d a), from pt₆_pt h₄,
  have h₆ : pt e f (pt d a (pt c b b)), from pt₃_pt (pt₂_pt h₅),
  have h₇ : pt e f (pt (pt d a c) b b), from pt₆_pt h₆,
  have h₈ : pt e f (pt d a c), from pt₅_pt h₇,
  show pt e f (pt a c d), from pt₃_pt (pt₂_pt h₈)
```

- pt'''_{11}

```
lemma pt₁₁''' {a b c d e f : Prop} (h₁ : pt e f (pt (pt a b c) a d))
  : pt e f (pt b c d) :=
  have h₂ : pt e f (pt d a (pt a b c)), from pt₂_pt (pt₃_pt (pt₂_pt h₁)),
  have h₃ : pt e f (pt (pt d a a) b c), from pt₆_pt h₂,
  have h₄ : pt e f (pt b c (pt d a a)), from pt₃_pt (pt₂_pt h₃),
  have h₅ : pt e f (pt (pt b c d) a a), from pt₆_pt h₄,
  show pt e f (pt b c d), from pt₅_pt h₅
```

- pt_{11}

```
theorem pt₁₁ {a b c d : Prop}
  (h₁ : pt (pt (pt a b c) a d) (pt (pt a b c) b d) (pt (pt a b c) c d)) : d :=
  have h₂ : pt (pt (pt a b c) a d) (pt (pt a b c) b d) (pt a b d), from pt₁₁' h₁,
```

```
  have h₃ : pt (pt (pt a b c) a d) (pt a b d) (pt (pt a b c) b d), from pt₃ h₂,
  have h₄ : pt (pt (pt a b c) a d) (pt a b d) (pt a c d), from pt₁₁'' h₃,
  have h₅ : pt (pt a b d) (pt (pt a b c) a d) (pt a c d), from pt₂ h₄,
  have h₆ : pt (pt a b d) (pt a c d) (pt (pt a b c) a d) , from pt₃ h₅,
  have h₇ : pt (pt a b d) (pt a c d) (pt b c d) , from pt₁₁''' h₆,
  have h₈ : pt a b (pt d (pt a c d) (pt b c d)) , from pt₇ h₇,
  have h₉ : pt a b (pt (pt a c d) d (pt b c d)) , from pt₂_pt h₈,
  have h₁₀ : pt (pt a b (pt a c d)) d (pt b c d) , from pt₆ h₉,
  have h₁₁ : pt (pt a b (pt a c d)) d (pt b d c) , from pt₃_pt h₁₀,
  have h₁₂ : pt (pt a b (pt a c d)) d (pt d b c) , from pt₂_pt h₁₁,
  have h₁₃ : pt (pt (pt a b (pt a c d)) d d) b c, from pt₆ h₁₂,
  have h₁₄ : pt b c (pt (pt a b (pt a c d)) d d), from pt₃ (pt₂ h₁₃),
  have h₁₅ : pt b c (pt a b (pt a c d)), from pt₅_pt h₁₄,
  have h₁₆ : pt b c (pt b a (pt a c d)), from pt₂_pt h₁₅,
  have h₁₇ : pt (pt b c b) a (pt a c d), from pt₆ h₁₆,
  have h₁₈ : pt a (pt a c d) (pt b c b), from pt₃ (pt₂ h₁₇),
  have h₁₉ : pt a (pt a c d) (pt c b b), from pt₂_pt h₁₈,
  have h₂₀ : pt a (pt a c d) c, from pt₅_pt h₁₉,
  have h₂₁ : pt (pt a c d) a c, from pt₂ h₂₀,
  show d, from pt₈ h₂₁
```

- pt_{12}

```
theorem pt₁₂ {a b c d e : Prop} (h₁ : pt a b c) (h₂ : d) (h₃ : e)
  : pt a b (pt c d e) :=
  have h₄ : pt (pt a b c) d e, from pt₁ h₁ h₂ h₃,
  show pt a b (pt c d e), from pt₇ h₄
```

- pt_{13}

```
theorem pt₁₃ {a b c d e : Prop} (h₁ : pt a b c) (h₂ : pt a b d) (h₃ : e)
  : (pt c d e) :=
  have h₄ : pt a b (pt c (pt a b d) e), from pt₁₂ h₁ h₂ h₃,
  have h₅ : pt a b (pt (pt a b d) c e), from pt₂_pt h₄,
  have h₆ : pt (pt a b (pt a b d)) c e, from pt₆ h₅,
  have h₇ : pt c e (pt a b (pt a b d)), from pt₃ (pt₂ h₆),
  have h₈ : pt c e (pt (pt a b d) a b), from pt₂_pt (pt₃_pt h₇),
  have h₉ : pt c e (pt b d b), from pt₁₁''' h₈,
  have h₁₀ : pt c e (pt d b b), from pt₂_pt h₉,
  have h₁₁ : pt c e d, from pt₅_pt h₁₀,
  show pt c d e, from pt₃ h₁₁
```

- pt_{14}

```
theorem pt₁₄ {a b c d e : Prop} (h₁ : pt a b c) (h₂ : pt a b d) (h₃ : pt a b e)
  : pt a b (pt c d e) :=
  have h₄ : pt c d (pt a b e), from pt₁₃ h₁ h₂ h₃,
  have h₅ : pt c d (pt e a b), from pt₂_pt (pt₃_pt h₄),
  have h₅ : pt (pt c d e) a b, from pt₆ h₅,
  show pt a b (pt c d e), from pt₃ (pt₂ h₅)
```

- pt_4'

```
theorem pt₄' {a b c : Prop} (h₁ : a) : pt (pt a b c) b c :=
  have h₂ : pt a b b, from pt₄ h₁,
  have h₃ : pt (pt a b b) c c, from pt₄ h₂,
  have h₄ : pt  a b (pt b c c), from pt₇ h₃,
  have h₅ : pt  a b (pt c b c), from pt₂_pt h₄,
  show pt (pt a b c) b c, from pt₆ h₅
```

□

We now proceed to prove what we call the monotonicity property m_{pt} and the deduction theorem δ_{pt}, using them in the sequel to to prove the completeness of $\mathscr{B}_{\mathsf{pt}}$ with respect to $\mathscr{Q}_{\mathsf{pt}}$.

Lemma 4.9.3. *The following property holds for* $\vdash_{\mathscr{B}_{\mathsf{pt}}}$*:*

$$\begin{aligned} &\text{for all } \Gamma \cup \{\mathrm{A}, \mathrm{B}, \mathrm{C}\} \subseteq L_{\mathsf{pt}}. \text{ if } \mathrm{A} \in \Gamma \text{ and } \Gamma, \mathrm{B} \vdash_{\mathscr{B}_{\mathsf{pt}}} \mathrm{C} \\ &\qquad \text{then } \Gamma \vdash_{\mathscr{B}_{\mathsf{pt}}} \mathsf{pt}(\mathrm{A}, \mathrm{B}, \mathrm{C}) \text{ or } \Gamma \vdash_{\mathscr{B}_{\mathsf{pt}}} \mathrm{C} \end{aligned} \tag{m_{pt}}$$

Proof. Let $\Gamma \cup \{\mathrm{A}, \mathrm{B}, \mathrm{C}\} \subseteq L_{\mathsf{pt}}$ and suppose that $\mathrm{A} \in \Gamma$ and $\Gamma, \mathrm{B} \vdash_{\mathscr{B}_{\mathsf{pt}}} \mathrm{C}$. Suppose that $\mathrm{C}_1, \ldots, \mathrm{C}_n = \mathrm{C}$ is an n-long sequence that witnesses the given consecution. Consider the property $P(i)$ given by the statement "$\Gamma \vdash_{\mathscr{B}_{\mathsf{pt}}} \mathsf{pt}(\mathrm{A}, \mathrm{B}, \mathrm{C}_i)$ or else $\Gamma \vdash_{\mathscr{B}_{\mathsf{pt}}} \mathrm{C}_i$". We work by induction on that derivation to show $P(k)$, for all $1 \leq k \leq n$, culminating in the desired conclusion when $k = n$. In this way, for the base case, there are two possibilities (since no axioms are available): (i) $\mathrm{C}_1 \in \Gamma$, in which case $\Gamma \vdash_{\mathscr{B}_{\mathsf{pt}}} \mathrm{C}_1$, or (ii) $\mathrm{C}_1 = \mathrm{B}$, in which case, by rule pt_4, from the assumption that $\mathrm{A} \in \Gamma$, we get $\Gamma \vdash_{\mathscr{B}_{\mathsf{pt}}} \mathsf{pt}(\mathrm{A}, \mathrm{B}, \mathrm{B})$, i.e. $\Gamma \vdash_{\mathscr{B}_{\mathsf{pt}}} \mathsf{pt}(\mathrm{A}, \mathrm{B}, \mathrm{C}_1)$. For the inductive step, given some $k > 1$, suppose that $P(i)$ holds for all $1 \leq i < k$, so it remains to prove $P(k)$. For that, there are three possibilities, two of them are those of the base case — already proven, then — and the third is that C_k follows from the instance $\langle \mathrm{C}_{k_1}, \ldots, \mathrm{C}_{k_m}, \mathrm{C}_k \rangle$, $k_j < k$, for all $1 \leq j \leq m$, of some primitive m-ary rule of $\mathscr{B}_{\mathsf{pt}}$. First, suppose that such rule is pt_1. Then, $\mathrm{C}_k = \mathsf{pt}(\mathrm{C}_{k_1}, \mathrm{C}_{k_2}, \mathrm{C}_{k_3})$. By the induction hypothesis, either $\Gamma \vdash_{\mathscr{B}_{\mathsf{pt}}} \mathsf{pt}(\mathrm{A}, \mathrm{B}, \mathrm{C}_{k_j})$ or $\Gamma \vdash_{\mathscr{B}_{\mathsf{pt}}} \mathrm{C}_{k_j}$, for each $1 \leq j \leq 3$. If the second case holds for them all, then simply use pt_1 to get $\Gamma \vdash_{\mathscr{B}_{\mathsf{pt}}} \mathrm{C}_k$. Otherwise, rules pt_{12}, pt_{13} and pt_{14} guarantee that, in any possible combination of the remaining cases, $\Gamma \vdash_{\mathscr{B}_{\mathsf{pt}}} \mathsf{pt}(\mathrm{A}, \mathrm{B}, \mathrm{C}_k)$ or else $\Gamma \vdash_{\mathscr{B}_{\mathsf{pt}}} \mathsf{pt}(\mathrm{C}_{k_1}, \mathrm{C}_{k_2}, \mathrm{C}_{k_3})$. Now, suppose that C_k follows from the one of the other rules, say pt_i, for some $2 \leq \mathsf{i} \leq 6$. Since they are all unary, there are only two cases, by the induction hypothesis: either $\Gamma \vdash_{\mathscr{B}_{\mathsf{pt}}} \mathsf{pt}(\mathrm{A}, \mathrm{B}, \mathrm{C}_{k_1})$ or $\Gamma \vdash_{\mathscr{B}_{\mathsf{pt}}} \mathrm{C}_{k_1}$, for C_{k_1} the formula from which C_k follows. In the first case, if the derivation occurs by the application of the rule, then use the derived rule $\mathsf{pt}_\mathsf{i}^{\mathsf{pt}}$ to get $\Gamma \vdash_{\mathscr{B}_{\mathsf{pt}}} \mathsf{pt}(\mathrm{A}, \mathrm{B}, \mathrm{C}_k)$. In the other case, apply the same rule pt_i to get $\Gamma \vdash_{\mathscr{B}_{\mathsf{pt}}} \mathrm{C}_k$. □

Theorem 4.9.4. *The following property holds for* $\vdash_{\mathscr{B}_{\mathsf{pt}}}$*:*

$$\begin{aligned} &\text{for all } \Gamma \cup \{\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}\} \subseteq L_{\mathsf{pt}}. \\ &\text{if } \Gamma, \mathrm{A} \vdash_{\mathscr{B}_{\mathsf{pt}}} \mathrm{D} \text{ and } \Gamma, \mathrm{B} \vdash_{\mathscr{B}_{\mathsf{pt}}} \mathrm{D} \text{ and } \Gamma, \mathrm{C} \vdash_{\mathscr{B}_{\mathsf{pt}}} \mathrm{D} \\ &\text{then } \Gamma, \mathsf{pt}(\mathrm{A}, \mathrm{B}, \mathrm{C}) \vdash_{\mathscr{B}_{\mathsf{pt}}} \mathrm{D} \end{aligned} \qquad (\delta_{\mathsf{pt}})$$

Proof. Let $\Gamma' = \Gamma \cup \{\mathsf{pt}(\mathrm{A}, \mathrm{B}, \mathrm{C})\}$, and suppose that these three consecutions hold: $\Gamma, \mathrm{A} \vdash_{\mathscr{B}_{\mathsf{pt}}} \mathrm{D}$, $\Gamma, \mathrm{B} \vdash_{\mathscr{B}_{\mathsf{pt}}} \mathrm{D}$, and $\Gamma, \mathrm{C} \vdash_{\mathscr{B}_{\mathsf{pt}}} \mathrm{D}$. By Lemma 4.9.3, considering each one of these assumptions, we get, respectively, $\Gamma' \vdash_{\mathscr{B}_{\mathsf{pt}}} \mathsf{pt}(\mathsf{pt}(\mathrm{A}, \mathrm{B}, \mathrm{C}), \mathrm{A}, \mathrm{D})$, $\Gamma' \vdash_{\mathscr{B}_{\mathsf{pt}}} \mathsf{pt}(\mathsf{pt}(\mathrm{A}, \mathrm{B}, \mathrm{C}), \mathrm{B}, \mathrm{D})$, and $\Gamma' \vdash_{\mathscr{B}_{\mathsf{pt}}} \mathsf{pt}(\mathsf{pt}(\mathrm{A}, \mathrm{B}, \mathrm{C}), \mathrm{C}, \mathrm{D})$ (abbreviate the right-hand sides of these consecutions by A', B' and C', respectively) or else $\Gamma' \vdash_{\mathscr{B}_{\mathsf{pt}}} \mathrm{D}$. The latter case gives precisely the desired result. For the other cases, notice that, by pt_1, we have $\Gamma' \vdash_{\mathscr{B}_{\mathsf{pt}}} \mathsf{pt}(\mathrm{A}', \mathrm{B}', \mathrm{C}')$, which, by the derived rule pt_{11}, gives $\Gamma' \vdash_{\mathscr{B}_{\mathsf{pt}}} \mathrm{D}$. □

Theorem 4.9.5. *The calculus* $\mathscr{B}_{\mathsf{pt}}$ *is complete with respect to the matrix* $\mathbb{2}_{\mathsf{pt}}$.

Proof. Following the procedure presented in Section 2.7, let $\Gamma \cup \{\mathrm{Z}\} \subseteq L_{\mathsf{pt}}$ and take the Z-maximal theory $\Gamma^+ \supseteq \Gamma$ via the Lindenbaum-Asser Lemma. The inspection of the truth-table for pt in $\mathbb{2}_{\mathsf{pt}}$ reveals the completeness property that needs to be proved:

$$\begin{aligned} \mathsf{pt}(\mathrm{A}, \mathrm{B}, \mathrm{C}) \in \Gamma^+ \text{ iff } &(\mathrm{A}, \mathrm{B}, \mathrm{C} \in \Gamma^+) \text{ or} \\ &(\mathrm{A} \in \Gamma^+ \text{ and } \mathrm{B}, \mathrm{C} \notin \Gamma^+) \text{ or} \\ &(\mathrm{B} \in \Gamma^+ \text{ and } \mathrm{A}, \mathrm{C} \notin \Gamma^+) \text{ or} \\ &(\mathrm{C} \in \Gamma^+ \text{ and } \mathrm{A}, \mathrm{B} \notin \Gamma^+) \end{aligned} \qquad (\mathsf{pt})$$

From the left to the right, suppose that $\mathsf{pt}(A, B, C) \in \Gamma^+$, thus (a): $\Gamma^+ \vdash_{\mathscr{B}_{\mathsf{pt}}} \mathsf{pt}(A, B, C)$. First of all, suppose, for the sake of contradiction, that $A, B, C \notin \Gamma^+$. Then, by Corollary 2.6.1.1, $\Gamma^+, A \vdash_{\mathscr{B}_{\mathsf{pt}}} Z$, $\Gamma^+, B \vdash_{\mathscr{B}_{\mathsf{pt}}} Z$ and $\Gamma^+, C \vdash_{\mathscr{B}_{\mathsf{pt}}} Z$; hence, by δ_{pt}, we get $\Gamma^+, \mathsf{pt}(A, B, C) \vdash_{\mathscr{B}_{\mathsf{pt}}} Z$, yielding, considering the consecution (a), the absurd $\Gamma^+ \vdash_{\mathscr{B}_{\mathsf{pt}}} Z$ by (T). We proceed by proving that it is not the case that any two of the formulas A, B and C can be in Γ^+ while the other is not. Consider the case in which $A, B \in \Gamma^+$ but $C \notin \Gamma^+$. Then, in view of (a), we get, by pt_1, $\Gamma^+ \vdash_{\mathscr{B}_{\mathsf{pt}}} \mathsf{pt}(\mathsf{pt}(A, B, C), A, B)$, which yields, by pt_8, $\Gamma^+ \vdash_{\mathscr{B}_{\mathsf{pt}}} C$. This, together with the assumption that $C \notin \Gamma^+$ (and thus $\Gamma^+, C \vdash_{\mathscr{B}_{\mathsf{pt}}} Z$) gives the absurd $\Gamma^+ \vdash_{\mathscr{B}_{\mathsf{pt}}} Z$ by (T). The cases $A, C \in \Gamma^+$ and $B \in \Gamma^+$, and $B, C \in \Gamma^+$ and $A \in \Gamma^+$ are handled in the same way, but using rules pt_9 and pt_{10} respectively, instead of pt_8. To finish this proof, consider the following two cases: either $A, B, C \in \Gamma^+$ or not. The first situation gives directly the desired result. The second case, considering the fact just proved, has as possibilities only the cases pursued to finish this proof, which must hold because we showed that at least one of the formulas A, B and C must be in Γ^+.

From the right to the left, suppose that $A, B, C \in \Gamma^+$, then $\Gamma^+ \vdash_{\mathscr{B}_{\mathsf{pt}}} \mathsf{pt}(A, B, C)$, by pt_1. Now, suppose that $A \in \Gamma^+$ and $B, C \notin \Gamma^+$, then $\Gamma^+, B \vdash_{\mathscr{B}_{\mathsf{pt}}} Z$ and $\Gamma^+, C \vdash_{\mathscr{B}_{\mathsf{pt}}} Z$. We proceed by contradiction: assume that $\mathsf{pt}(A, B, C) \notin \Gamma^+$, meaning that $\Gamma^+, \mathsf{pt}(A, B, C) \vdash_{\mathscr{B}_{\mathsf{pt}}} Z$. Then, by δ_{pt}, the consecution (a): $\Gamma^+, \mathsf{pt}(\mathsf{pt}(A, B, C), B, C) \vdash_{\mathscr{B}_{\mathsf{pt}}} Z$ follows. From $\Gamma^+ \vdash_{\mathscr{B}_{\mathsf{pt}}} A$, by pt'_4, we get $\Gamma^+ \vdash_{\mathscr{B}_{\mathsf{pt}}} \mathsf{pt}(\mathsf{pt}(A, B, C), B, C)$, which, together with (a), derives the absurd $\Gamma^+ \vdash_{\mathscr{B}_{\mathsf{pt}}} Z$ by (T). The proofs for the other two cases are analogous. □

Remark 4.9.1. Notice that a sufficient condition for the preservation of the property m_{pt}, and thus the completeness property (pt), in any expansion of the calculus $\mathscr{B}_{\mathsf{pt}}$ by non-nullary rules is that, for any of the new rules, say r, its lifted version r^{pt} is derivable in the expanded calculus.

According to Rautenberg [11, p. 332], the fragment $\mathcal{B}_{\mathsf{pt},\bot}$ is axiomatized by merging the calculi $\mathscr{B}_{\mathsf{pt}}$ and $\mathscr{B}_{\bot}$, as presented below, because m_{pt} and thus δ_{pt} are preserved after this combination. This preservation, according to the author, occurs because the particular case of m_{pt} in which $B = \bot$ holds for the resulting calculus. We highlight here that, although we see how this specialization implies m_{pt}, we could not verify this result and so a further investigation is necessary for this specific case.

Hilbert Calculus 25. $\mathscr{B}_{\mathsf{pt},\bot}$

$\mathscr{B}_{\mathsf{pt}}$ $\mathscr{B}_{\bot}$

The expansions $\mathcal{B}_{\mathsf{pt},\top}$ and $\mathcal{B}_{\mathsf{pt},\bot,\top}$ are directly axiomatized, respectively, by the calculi below, in view of Corollary 2.8.4.1.

Hilbert Calculus 26. $\mathscr{B}_{\mathsf{pt},\top}$

$\mathscr{B}_{\mathsf{pt}}$ $\mathscr{B}_{\top}$

Hilbert Calculus 27. $\mathscr{B}_{\mathsf{pt},\bot,\top}$

$\mathscr{B}_{\mathsf{pt},\bot}$ $\mathscr{B}_{\top}$

4.10 $\mathcal{B}_{\mathsf{pt},\neg}$

We now deal with the fragment on the language containing only the connectives pt and $\neg$. The candidate calculus extends $\mathscr{B}_{\mathsf{pt}}$, presented in Section 4.9, by adding the rule of explosion and some interaction rules.

Hilbert Calculus 28. $\mathscr{B}_{\mathsf{pt},\neg}$

$\mathscr{B}_{\mathsf{pt}}$

$$\frac{A \quad \neg A}{B}\,\mathsf{n}_1 \qquad \frac{\neg\mathsf{pt}(A,B,C)}{\mathsf{pt}(\neg A,B,C)}\,\mathsf{ptn}_1 \qquad \frac{\mathsf{pt}(\neg A,B,C)}{\neg\mathsf{pt}(A,B,C)}\,\mathsf{ptn}_2 \qquad \frac{\mathsf{pt}(\neg A,B,C)}{\mathsf{pt}(A,\neg B,C)}\,\mathsf{ptn}_3$$

The axiomatization above differs from the one presented in [11], which consists in merging the calculi $\mathscr{B}_{\mathsf{pt}}$ and $\mathscr{B}_{\neg}$, plus rules ptn_1 and ptn_2. Besides having one less rule, our calculus eases the derivation of the rules necessary to prove completeness. We proceed now by showing that $\mathscr{B}_{\mathsf{pt},\neg}$ is sound with respect to $\mathcal{B}_{\mathsf{pt},\neg}$.

Theorem 4.10.1. *The calculus $\mathscr{B}_{\mathsf{pt},\neg}$ is sound with respect to the matrix $\mathbb{2}_{\mathsf{pt},\neg}$.*

Proof. We know from Theorem 4.8.1 and Theorem 4.9.1 that n_1 and the rules of $\mathscr{B}_{\mathsf{pt}}$ are sound with respect to $\mathfrak{2}_{\mathsf{pt},\neg}$. Now, let v be some $\mathfrak{2}_{\mathsf{pt},\neg}$-valuation. For ptn_1, suppose that $v(\neg\mathsf{pt}(\mathrm{A},\mathrm{B},\mathrm{C})) = 1$, then $v(\mathsf{pt}(\mathrm{A},\mathrm{B},\mathrm{C})) = 0$ and we can consider the following cases:

- if $v(\mathrm{A}) = 0$, $v(\mathrm{B}) = 0$ and $v(\mathrm{C}) = 0$, then $v(\neg\mathrm{A}) = 1$ and $v(\mathsf{pt}(\neg\mathrm{A},\mathrm{B},\mathrm{C})) = \mathsf{pt}^{\mathfrak{2}_{\mathsf{pt},\neg}}(v(\neg\mathrm{A}),v(\mathrm{B}),v(\mathrm{C})) = \mathsf{pt}^{\mathfrak{2}_{\mathsf{pt},\neg}}(1,0,0) = 1$.

- if $v(\mathrm{A}) = 1$, $v(\mathrm{B}) = 1$ and $v(\mathrm{C}) = 0$, then $v(\neg\mathrm{A}) = 0$ and $v(\mathsf{pt}(\neg\mathrm{A},\mathrm{B},\mathrm{C})) = \mathsf{pt}^{\mathfrak{2}_{\mathsf{pt},\neg}}(v(\neg\mathrm{A}),v(\mathrm{B}),v(\mathrm{C})) = \mathsf{pt}^{\mathfrak{2}_{\mathsf{pt},\neg}}(0,1,0) = 1$.

- if $v(\mathrm{A}) = 1$, $v(\mathrm{B}) = 0$ and $v(\mathrm{C}) = 1$, then $v(\neg\mathrm{A}) = 0$ and $v(\mathsf{pt}(\neg\mathrm{A},\mathrm{B},\mathrm{C})) = \mathsf{pt}^{\mathfrak{2}_{\mathsf{pt},\neg}}(v(\neg\mathrm{A}),v(\mathrm{B}),v(\mathrm{C})) = \mathsf{pt}^{\mathfrak{2}_{\mathsf{pt},\neg}}(0,0,1) = 1$.

- if $v(\mathrm{A}) = 0$, $v(\mathrm{B}) = 1$ and $v(\mathrm{C}) = 1$, then $v(\neg\mathrm{A}) = 1$ and $v(\mathsf{pt}(\neg\mathrm{A},\mathrm{B},\mathrm{C})) = \mathsf{pt}^{\mathfrak{2}_{\mathsf{pt},\neg}}(v(\neg\mathrm{A}),v(\mathrm{B}),v(\mathrm{C})) = \mathsf{pt}^{\mathfrak{2}_{\mathsf{pt},\neg}}(1,1,1) = 1$.

For ptn_2, suppose that $v(\mathsf{pt}(\neg\mathrm{A},\mathrm{B},\mathrm{C})) = 1$ and consider the following cases:

- if $v(\neg\mathrm{A}) = 1$, $v(\mathrm{B}) = 1$ and $v(\mathrm{C}) = 1$, then $v(\mathrm{A}) = 0$ and $v(\mathsf{pt}(\mathrm{A},\mathrm{B},\mathrm{C})) = \mathsf{pt}^{\mathfrak{2}_{\mathsf{pt},\neg}}(v(\mathrm{A}),v(\mathrm{B}),v(\mathrm{C})) = \mathsf{pt}^{\mathfrak{2}_{\mathsf{pt},\neg}}(0,1,1) = 0$, whose negation gives 1.

- if $v(\neg\mathrm{A}) = 1$, $v(\mathrm{B}) = 0$ and $v(\mathrm{C}) = 0$, then $v(\mathrm{A}) = 0$ and $v(\mathsf{pt}(\mathrm{A},\mathrm{B},\mathrm{C})) = \mathsf{pt}^{\mathfrak{2}_{\mathsf{pt},\neg}}(v(\mathrm{A}),v(\mathrm{B}),v(\mathrm{C})) = \mathsf{pt}^{\mathfrak{2}_{\mathsf{pt},\neg}}(0,0,0) = 0$, whose negation gives 1.

- if $v(\neg\mathrm{A}) = 0$, $v(\mathrm{B}) = 0$ and $v(\mathrm{C}) = 1$, then $v(\mathrm{A}) = 1$ and $v(\mathsf{pt}(\mathrm{A},\mathrm{B},\mathrm{C})) = \mathsf{pt}^{\mathfrak{2}_{\mathsf{pt},\neg}}(v(\mathrm{A}),v(\mathrm{B}),v(\mathrm{C})) = \mathsf{pt}^{\mathfrak{2}_{\mathsf{pt},\neg}}(1,0,1) = 0$, whose negation gives 1.

- if $v(\neg\mathrm{A}) = 0$, $v(\mathrm{B}) = 1$ and $v(\mathrm{C}) = 0$, then $v(\mathrm{A}) = 1$ and $v(\mathsf{pt}(\mathrm{A},\mathrm{B},\mathrm{C})) = \mathsf{pt}^{\mathfrak{2}_{\mathsf{pt},\neg}}(v(\mathrm{A}),v(\mathrm{B}),v(\mathrm{C})) = \mathsf{pt}^{\mathfrak{2}_{\mathsf{pt},\neg}}(1,1,0) = 0$, whose negation gives 1.

Finally, for ptn_3, suppose that $v(\mathsf{pt}(\neg\mathrm{A},\mathrm{B},\mathrm{C})) = 1$ and consider the following cases:

- if $v(\neg\mathrm{A}) = 1$, $v(\mathrm{B}) = 1$ and $v(\mathrm{C}) = 1$, then $v(\mathrm{A}) = 0$ and $v(\neg\mathrm{B}) = 0$, so $v(\mathsf{pt}(\mathrm{A},\neg\mathrm{B},\mathrm{C})) = \mathsf{pt}^{\mathfrak{2}_{\mathsf{pt},\neg}}(v(\mathrm{A}),v(\neg\mathrm{B}),v(\mathrm{C})) = \mathsf{pt}^{\mathfrak{2}_{\mathsf{pt},\neg}}(0,0,1) = 1$;

- if $v(\neg\mathrm{A}) = 1$, $v(\mathrm{B}) = 0$ and $v(\mathrm{C}) = 0$, then $v(\mathrm{A}) = 0$ and $v(\neg\mathrm{B}) = 1$, so $v(\mathsf{pt}(\mathrm{A},\neg\mathrm{B},\mathrm{C})) = \mathsf{pt}^{\mathfrak{2}_{\mathsf{pt},\neg}}(v(\mathrm{A}),v(\neg\mathrm{B}),v(\mathrm{C})) = \mathsf{pt}^{\mathfrak{2}_{\mathsf{pt},\neg}}(0,1,0) = 1$;

- if $v(\neg\mathrm{A}) = 0$, $v(\mathrm{B}) = 1$ and $v(\mathrm{C}) = 0$, then $v(\mathrm{A}) = 1$ and $v(\neg\mathrm{B}) = 0$, so $v(\mathsf{pt}(\mathrm{A},\neg\mathrm{B},\mathrm{C})) = \mathsf{pt}^{\mathfrak{2}_{\mathsf{pt},\neg}}(v(\mathrm{A}),v(\neg\mathrm{B}),v(\mathrm{C})) = \mathsf{pt}^{\mathfrak{2}_{\mathsf{pt},\neg}}(1,0,0) = 1$;

- if $v(\neg A) = 0$, $v(B) = 0$ and $v(C) = 1$, then $v(A) = 1$ and $v(\neg B) = 1$, so $v(\mathsf{pt}(A, \neg B, C)) = \mathsf{pt}^{\mathcal{Q}_{\mathsf{pt},\neg}}(v(A), v(\neg B), v(C)) = \mathsf{pt}^{\mathcal{Q}_{\mathsf{pt},\neg}}(1, 1, 1) = 1$.

□

The next result presents the derivation of the negated versions of some rules of $\mathscr{B}_{\mathsf{pt}}$ (same premisses and conclusions, but with a negation in front), which will be very useful in proving the pt-lifted versions of the rules n_1 and ptn_i, for each $1 \leq \mathsf{i} \leq 3$, the necessary ingredients to guarantee that the completeness property (pt) holds in $\mathscr{B}_{\mathsf{pt},\neg}$, according to Remark 4.9.1.

Lemma 4.10.2. *The following rules are derivable in* $\mathscr{B}_{\mathsf{pt},\neg}$*:*

$$\frac{\neg\mathsf{pt}(A, B, C)}{\neg\mathsf{pt}(B, A, C)}\ \mathsf{pt}_2^\neg$$

$$\frac{\neg\mathsf{pt}(A, B, C)}{\neg\mathsf{pt}(A, C, B)}\ \mathsf{pt}_3^\neg$$

$$\frac{\neg A}{\neg\mathsf{pt}(A, B, B)}\ \mathsf{pt}_4^\neg$$

$$\frac{\neg\mathsf{pt}(A, B, B)}{\neg A}\ \mathsf{pt}_5^\neg$$

$$\frac{\neg\mathsf{pt}(A, B, \mathsf{pt}(C, D, E))}{\neg\mathsf{pt}(\mathsf{pt}(A, B, C), D, E)}\ \mathsf{pt}_6^\neg$$

$$\frac{\neg\mathsf{pt}(\mathsf{pt}(A, B, C), D, E)}{\neg\mathsf{pt}(A, B, \mathsf{pt}(C, D, E))}\ \mathsf{pt}_7^\neg$$

$$\frac{\mathsf{pt}(C, D, A) \quad \mathsf{pt}(C, D, \neg A)}{\mathsf{pt}(C, D, B)}\ \mathsf{n}_1^{\mathsf{pt}}$$

$$\frac{\mathsf{pt}(D, E, \neg\mathsf{pt}(A, B, C))}{\mathsf{pt}(D, E, \mathsf{pt}(\neg A, B, C))}\ \mathsf{ptn}_1^{\mathsf{pt}}$$

$$\frac{\mathsf{pt}(D, E, \mathsf{pt}(\neg A, B, C))}{\mathsf{pt}(D, E, \neg\mathsf{pt}(A, B, C))}\ \mathsf{ptn}_2^{\mathsf{pt}}$$

$$\frac{\mathsf{pt}(D, E, \mathsf{pt}(\neg A, B, C))}{\mathsf{pt}(D, E, \mathsf{pt}(A, \neg B, C))}\ \mathsf{ptn}_3^{\mathsf{pt}}$$

Proof. The formally verified derivation of each rule is presented below, following what was explained in Chapter 3.

- $\mathsf{pt}_2^\neg$

```
theorem pt₂_neg {a b c : Prop} (h₁ : neg (pt a b c)) : neg (pt b a c) :=
  have h₂ : pt (neg a) b c, from ptn₁ h₁,
```

```
  have h₃ : pt a (neg b) c, from ptn₃ h₂,
  have h₄ : pt (neg b) a c, from pt.pt₂ h₃,
  show neg (pt b a c), from ptn₂ h₄
```

- $pt_3^{\neg}$

```
theorem pt₃_neg {a b c : Prop} (h₁ : neg (pt a b c)) : neg (pt a c b) :=
  have h₂ : pt (neg a) b c, from ptn₁ h₁,
  have h₃ : pt (neg a) c b, from pt.pt₃ h₂,
  show neg (pt a c b), from ptn₂ h₃
```

- $pt_4^{\neg}$

```
theorem pt₄_neg {a b : Prop} (h₁ : neg a) : neg (pt a b b) :=
  have h₂ : pt (neg a) b b, from pt.pt₄ h₁,
  show neg (pt a b b), from ptn₂ h₂
```

- $pt_5^{\neg}$

```
theorem pt₅_neg {a b : Prop} (h₁ : neg (pt a b b)) : neg a :=
  have h₂ : pt (neg a) b b, from ptn₁ h₁,
  show neg a, from pt.pt₅ h₂
```

- $pt_6^{\neg}$

```
theorem pt₆_neg {a b c d e : Prop} (h₁ : neg (pt a b (pt c d e))) : neg (pt (pt a b c) d e) :=
  have h₂ : pt (neg a) b (pt c d e), from ptn₁ h₁,
  have h₃ : pt (pt (neg a) b c) d e, from pt.pt₆ h₂,
  have h₄ : pt d e (pt (neg a) b c), from pt.pt₃ (pt.pt₂ h₃),
  have h₅ : pt (pt d e (neg a)) b c, from pt.pt₆ h₄,
  have h₆ : pt b c (pt d e (neg a)), from pt.pt₃ (pt.pt₂ h₅),
  have h₇ : pt b c (pt (neg a) d e), from pt.pt₂_ast (pt.pt₃_ast h₆),
  have h₈ : pt (pt (neg a) d e) b c, from pt.pt₂ (pt.pt₃ h₇),
  have h₉ : pt (neg a) d (pt e b c), from pt.pt₇ h₈,
  have h₁₀ : neg (pt a d (pt e b c)), from ptn₂ h₉,
  have h₁₁ : neg (pt d a (pt e b c)), from pt₂_neg h₁₀,
  have h₁₂ : pt (neg d) a (pt e b c), from ptn₁ h₁₁,
  have h₁₃ : pt (pt (neg d) a e) b c, from pt.pt₆ h₁₂,
  have h₁₄ : pt b c (pt (neg d) a e), from pt.pt₃ (pt.pt₂ h₁₃),
  have h₁₅ : pt b c (pt a (neg d) e), from pt.pt₂_ast h₁₄,
```

```
have h₁₆ : pt (pt b c a) (neg d) e, from pt.pt₆ h₁₅,
have h₁₇ : pt (neg d) e (pt b c a), from pt.pt₃ (pt.pt₂ h₁₆),
have h₁₈ : pt (neg d) e (pt a b c), from pt.pt₂_ast (pt.pt₃_ast h₁₇),
have h₁₉ : neg (pt d e (pt a b c)), from ptn₂ h₁₈,
show neg (pt (pt a b c) d e), from pt₂_neg (pt₃_neg h₁₉)
```

- $pt_7^{\neg}$

```
theorem pt₇_neg {a b c d e : Prop} (h₁ : neg (pt (pt a b c) d e)) : neg (pt a b (pt c d e)) :=
  have h₂ : neg (pt d (pt a b c) e), from pt₂_neg h₁,
  have h₃ : neg (pt d e (pt a b c)), from pt₃_neg h₂,
  have h₄ : neg (pt (pt d e a) b c), from pt₆_neg h₃,
  have h₅ : neg (pt b (pt d e a) c), from pt₂_neg h₄,
  have h₆ : neg (pt b c (pt d e a)), from pt₃_neg h₅,
  have h₇ : neg (pt (pt b c d) e a), from pt₆_neg h₆,
  have h₈ : neg (pt e (pt b c d) a), from pt₂_neg h₇,
  have h₉ : neg (pt e a (pt b c d)), from pt₃_neg h₈,
  have h₁₀ : neg (pt (pt e a b) c d), from pt₆_neg h₉,
  have h₁₁ : neg (pt c (pt e a b) d), from pt₂_neg h₁₀,
  have h₁₂ : neg (pt c d (pt e a b)), from pt₃_neg h₁₁,
  have h₁₃ : neg (pt (pt c d e) a b), from pt₆_neg h₁₂,
  have h₁₄ : neg (pt a (pt c d e) b), from pt₂_neg h₁₃,
  show neg (pt a b (pt c d e)), from pt₃_neg h₁₄
```

- n_1^{pt}

```
theorem n₁_pt {a b c d : Prop} (h₁ : pt c d a) (h₂ : pt c d (neg a)) : pt c d b :=
  have h₃ : pt a c d, from pt.pt₂ (pt.pt₃ h₁),
  have h₄ : pt (neg a) c d, from pt.pt₂ (pt.pt₃ h₂),
  have h₅ : neg (pt a c d), from ptn₂ h₄,
  show pt c d b, from n₁ h₃ h₅
```

- ptn_1^{pt}

```
theorem ptn₁_pt {a b c d e : Prop} (h₁ : pt d e (neg (pt a b c)))
  : pt d e (pt (neg a) b c) :=
  have h₂ : pt (neg (pt a b c)) d e, from pt.pt₂ (pt.pt₃ h₁),
  have h₃ : neg (pt (pt a b c) d e), from ptn₂ h₂,
  have h₄ : neg (pt a b (pt c d e)), from pt₇_neg h₃,
  have h₅ : pt (neg a) b (pt c d e), from ptn₁ h₄,
  have h₆ : pt (pt (neg a) b c) d e, from pt.pt₆ h₅,
```

```
    show pt d e (pt (neg a) b c), from pt.pt₃ (pt.pt₂ h₆)
```

- $\mathsf{ptn}_2^{\mathsf{pt}}$

```
theorem ptn₂_pt {a b c d e : Prop} (h₁ : pt d e (pt (neg a) b c))
  : pt d e (neg (pt a b c)) :=
  have h₂ : pt (pt (neg a) b c) d e, from pt.pt₂ (pt.pt₃ h₁),
  have h₃ : pt (neg a) b (pt c d e), from pt.pt₇ h₂,
  have h₄ : neg (pt a b (pt c d e)), from ptn₂ h₃,
  have h₅ : neg (pt (pt a b c) d e), from pt₆_neg h₄,
  have h₆ : pt (neg (pt a b c)) d e, from ptn₁ h₅,
  show pt d e (neg (pt a b c)), from pt.pt₃ (pt.pt₂ h₆)
```

- $\mathsf{ptn}_3^{\mathsf{pt}}$

```
theorem ptn₃_pt {a b c d e : Prop} (h₁ : pt d e (neg (pt a b c)))
  : pt d e (pt a (neg b) c) :=
  have h₂ : pt (neg (pt a b c)) d e, from pt.pt₂ (pt.pt₃ h₁),
  have h₃ : neg (pt (pt a b c) d e), from ptn₂ h₂,
  have h₄ : neg (pt a b (pt c d e)), from pt₇_neg h₃,
  have h₅ : neg (pt b a (pt c d e)), from pt₂_neg h₄,
  have h₆ : pt (neg b) a (pt c d e), from ptn₁ h₅,
  have h₇ : pt a (neg b) (pt c d e), from pt.pt₂ h₆,
  have h₈ : pt (pt a (neg b) c) d e, from pt.pt₆ h₇,
  show pt d e (pt a (neg b) c), from pt.pt₃ (pt.pt₂ h₈)
```

□

Theorem 4.10.3. *The calculus $\mathscr{B}_{\mathsf{pt},\neg}$ is complete with respect to the matrix $\mathfrak{2}_{\mathsf{pt},\neg}$.*

Proof. According to the procedure presented in Section 2.7, we need to prove that the completeness properties $(\neg)$ and (pt) — introduced respectively in the proofs of Theorem 4.8.4 and Theorem 4.9.5 — hold in $\mathscr{B}_{\mathsf{pt},\neg}$. By Remark 4.9.1, the derivability of $\mathsf{n}_1^{\mathsf{pt}}$ and $\mathsf{ptn}_i^{\mathsf{pt}}$, for each $1 \leq i \leq 3$, implies that properties m_{pt} and (pt) hold in $\mathscr{B}_{\mathsf{pt},\neg}$. Now, remember that the completeness property for $\neg$ is $(\neg)$ $\neg A \in \Gamma^+$ iff $A \notin \Gamma^+$, for a Z-maximal $\Gamma^+ \supseteq \Gamma$, where $\Gamma \cup \{Z\} \subseteq L_{\mathsf{pt},\neg}$ and $\Gamma \nvdash_{\mathscr{B}_{\mathsf{pt},\neg}} Z$. The left-to-right direction follows because of rule n_1. The converse is more involving and to prove it we work by contradiction: suppose that $A \notin \Gamma^+$ and $\neg A \notin \Gamma^+$. Then, by Corollary 2.6.1.1, (a): $\Gamma^+, A \vdash_{\mathscr{B}_{\mathsf{pt},\neg}} Z$ and (b): $\Gamma^+, \neg A \vdash_{\mathscr{B}_{\mathsf{pt},\neg}} Z$.

Since $\mathcal{B}_{\mathsf{pt},\neg}$ has no tautologies, Lemma 2.6.2 allows us to take some $\mathrm{B} \in \Gamma^+$. Hence, by (a), (b) and m_{pt}, we get $\Gamma^+ \vdash_{\mathscr{B}_{\mathsf{pt},\neg}} \mathsf{pt}(\mathrm{B}, \mathrm{A}, \mathrm{Z})$ and $\Gamma^+ \vdash_{\mathscr{B}_{\mathsf{pt},\neg}} \mathsf{pt}(\mathrm{B}, \neg\mathrm{A}, \mathrm{Z})$ (since $\Gamma^+ \vdash_{\mathscr{B}_{\mathsf{pt},\neg}} \mathrm{Z}$ is not the case), yielding (c): $\Gamma^+ \vdash_{\mathscr{B}_{\mathsf{pt},\neg}} \mathsf{pt}(\mathrm{A}, \mathrm{B}, \mathrm{Z})$ and $\Gamma^+ \vdash_{\mathscr{B}_{\mathsf{pt},\neg}} \mathsf{pt}(\neg\mathrm{A}, \mathrm{B}, \mathrm{Z})$ by rule pt_2. The latter consecution, by ptn_2, gives (d): $\Gamma^+ \vdash_{\mathscr{B}_{\mathsf{pt},\neg}} \neg\mathsf{pt}(\mathrm{A}, \mathrm{B}, \mathrm{Z})$. Thus, by rule n_1, from (c) and (d), we have $\Gamma^+ \vdash_{\mathscr{B}_{\mathsf{pt},\neg}} \mathrm{Z}$, an absurd. □

4.11 $\mathcal{B}_{\mathsf{dc}}$

The classical connective dc may be defined from those in $\mathcal{B}$ by means of the translation $\mathbf{t}(\mathsf{dc}) = \lambda \mathrm{p}, \mathrm{q}, \mathrm{r}.(\mathrm{p} \wedge \mathrm{q}) \vee (\mathrm{q} \wedge \mathrm{r}) \vee (\mathrm{p} \wedge \mathrm{r})$. An inspection of the truth table of $\mathsf{dc}^{\mathbb{2}}$ reveals that $\mathsf{dc}^{\mathbb{2}}(x, y, z) = 1$ if, and only if, exactly two or all of the arguments x, y and z are 1. The calculus presented below is a candidate axiomatization for the fragment $\mathcal{B}_{\mathsf{dc}}$. In the sequel, we prove its soundness with respect to $\mathbb{2}_{\mathsf{dc}}$.

Hilbert Calculus 29. $\mathscr{B}_{\mathsf{dc}}$

$$\frac{\mathrm{A} \quad \mathrm{B}}{\mathsf{dc}(\mathrm{A}, \mathrm{B}, \mathrm{C})}\ \mathsf{dc}_1$$

$$\frac{\mathsf{dc}(\mathrm{B}, \mathrm{A}, \mathrm{A})}{\mathrm{A}}\ \mathsf{dc}_2$$

$$\frac{\mathrm{A}}{\mathsf{dc}(\mathrm{B}, \mathrm{A}, \mathrm{A})}\ \mathsf{dc}_3$$

$$\frac{\mathsf{dc}(\mathrm{D}, \mathrm{E}, \mathsf{dc}(\mathrm{A}, \mathrm{B}, \mathrm{C}))}{\mathsf{dc}(\mathrm{E}, \mathrm{D}, \mathsf{dc}(\mathrm{B}, \mathrm{A}, \mathrm{C}))}\ \mathsf{dc}_4$$

$$\frac{\mathsf{dc}(\mathrm{D}, \mathrm{E}, \mathsf{dc}(\mathrm{A}, \mathrm{B}, \mathrm{C}))}{\mathsf{dc}(\mathrm{E}, \mathrm{D}, \mathsf{dc}(\mathrm{A}, \mathrm{C}, \mathrm{B}))}\ \mathsf{dc}_5$$

$$\frac{\mathsf{dc}(\mathrm{F}, \mathrm{G}, \mathsf{dc}(\mathrm{D}, \mathrm{E}, \mathsf{dc}(\mathrm{A}, \mathrm{B}, \mathrm{C})))}{\mathsf{dc}(\mathrm{F}, \mathrm{G}, \mathsf{dc}(\mathsf{dc}(\mathrm{D}, \mathrm{E}, \mathrm{A}), \mathsf{dc}(\mathrm{D}, \mathrm{E}, \mathrm{B}), \mathrm{C}))}\ \mathsf{dc}_6$$

$$\frac{\mathsf{dc}(\mathrm{F}, \mathrm{G}, \mathsf{dc}(\mathsf{dc}(\mathrm{D}, \mathrm{E}, \mathrm{A}), \mathsf{dc}(\mathrm{D}, \mathrm{E}, \mathrm{B}), \mathrm{C}))}{\mathsf{dc}(\mathrm{F}, \mathrm{G}, \mathsf{dc}(\mathrm{D}, \mathrm{E}, \mathsf{dc}(\mathrm{A}, \mathrm{B}, \mathrm{C})))}\ \mathsf{dc}_7$$

Theorem 4.11.1. *The calculus $\mathscr{B}_{\mathsf{dc}}$ is sound with respect to the matrix $\mathbb{2}_{\mathsf{dc}}$.*

Proof. Let v be a $\mathbb{2}_{\mathsf{dc}}$-valuation. For rule dc_1, if $v(\mathrm{A}) = 1$ and $v(\mathrm{B}) = 1$, then $v(\mathsf{dc}(\mathrm{A}, \mathrm{B}, \mathrm{C})) = \mathsf{dc}^{\mathbb{2}_{\mathsf{dc}}}(v(\mathrm{A}), v(\mathrm{B}), v(\mathrm{C})) = \mathsf{dc}^{\mathbb{2}_{\mathsf{dc}}}(1, 1, v(\mathrm{C})) = 1$. For rule dc_2, if $v(\mathrm{A}) = 0$, then $v(\mathsf{dc}(\mathrm{B}, \mathrm{A}, \mathrm{A})) = \mathsf{dc}^{\mathbb{2}_{\mathsf{dc}}}(v(\mathrm{B}), v(\mathrm{A}), v(\mathrm{A})) = \mathsf{dc}^{\mathbb{2}_{\mathsf{dc}}}(v(\mathrm{B}), 0, 0) = 0$. The argument for

dc_3 is analogous to that for dc_1. For rules dc_4 and dc_5, it is enough to notice that permuting the components of $\mathsf{ad}(\mathrm{A},\mathrm{B},\mathrm{C})$ does not change its value under v. Finally, for unary rules dc_6 and dc_7 (one is the converse of the other), we can check that the two involved formulas are logically equivalent considering $\vdash_{\mathbb{2}}$, and thus also in $\vdash_{\mathbb{2}_{\mathsf{dc}}}$. □

In what follows, if r is an n-ary rule, with $n \in \omega$, let r^{dc}, the dc-lifted version of r, be the rule given by the set of instances $\langle \mathsf{dc}(\mathrm{C},\mathrm{D},\mathrm{A}_1),\ldots,\mathsf{dc}(\mathrm{C},\mathrm{D},\mathrm{A}_n),\mathsf{dc}(\mathrm{C},\mathrm{D},\mathrm{B})\rangle$, where $\langle \mathrm{A}_1,\ldots,\mathrm{A}_n,\mathrm{B}\rangle$ is an instance of r and $\mathrm{C},\mathrm{D} \in L_{\mathsf{dc}}$. The next lemma presents the derivability of some rules in $\mathscr{B}_{\mathsf{dc}}$, including the dc-lifted versions of its primitive rules, which lead to the completeness result with respect to $\mathbb{2}_{\mathsf{dc}}$, as we will see in the sequel.

Lemma 4.11.2. *The following rules are derivable in $\mathscr{B}_{\mathsf{dc}}$:*

$$\frac{\mathsf{dc}(\mathrm{A},\mathrm{B},\mathrm{C})}{\mathsf{dc}(\mathrm{B},\mathrm{A},\mathrm{C})}\ \mathsf{dc}_4'$$

$$\frac{\mathsf{dc}(\mathrm{A},\mathrm{B},\mathrm{C})}{\mathsf{dc}(\mathrm{A},\mathrm{C},\mathrm{B})}\ \mathsf{dc}_5'$$

$$\frac{\mathsf{dc}(\mathrm{D},\mathrm{E},\mathsf{dc}(\mathrm{A},\mathrm{B},\mathrm{C}))}{\mathsf{dc}(\mathsf{dc}(\mathrm{D},\mathrm{E},\mathrm{A}),\mathsf{dc}(\mathrm{D},\mathrm{E},\mathrm{B}),\mathrm{C})}\ \mathsf{dc}_6'$$

$$\frac{\mathsf{dc}(\mathsf{dc}(\mathrm{D},\mathrm{E},\mathrm{A}),\mathsf{dc}(\mathrm{D},\mathrm{E},\mathrm{B}),\mathrm{C})}{\mathsf{dc}(\mathrm{D},\mathrm{E},\mathsf{dc}(\mathrm{A},\mathrm{B},\mathrm{C}))}\ \mathsf{dc}_7'$$

$$\frac{\mathsf{dc}(\mathrm{D},\mathrm{E},\mathrm{A}) \quad \mathsf{dc}(\mathrm{D},\mathrm{E},\mathrm{B})}{\mathsf{dc}(\mathrm{D},\mathrm{E},\mathsf{dc}(\mathrm{A},\mathrm{B},\mathrm{C}))}\ \mathsf{dc}_1^{\mathsf{dc}}$$

$$\frac{\mathsf{dc}(\mathrm{C},\mathrm{D},\mathsf{dc}(\mathrm{B},\mathrm{A},\mathrm{A}))}{\mathsf{dc}(\mathrm{C},\mathrm{D},\mathrm{A})}\ \mathsf{dc}_2^{\mathsf{dc}}$$

$$\frac{\mathsf{dc}(\mathrm{C},\mathrm{D},\mathrm{A})}{\mathsf{dc}(\mathrm{C},\mathrm{D},\mathsf{dc}(\mathrm{B},\mathrm{A},\mathrm{A}))}\ \mathsf{dc}_3^{\mathsf{dc}}$$

$$\frac{\mathsf{dc}(\mathrm{F},\mathrm{G},\mathsf{dc}(\mathrm{D},\mathrm{E},\mathsf{dc}(\mathrm{A},\mathrm{B},\mathrm{C})))}{\mathsf{dc}(\mathrm{F},\mathrm{G},\mathsf{dc}(\mathrm{E},\mathrm{D},\mathsf{dc}(\mathrm{B},\mathrm{A},\mathrm{C})))}\ \mathsf{dc}_4^{\mathsf{dc}}$$

$$\frac{\mathsf{dc}(\mathrm{F},\mathrm{G},\mathsf{dc}(\mathrm{D},\mathrm{E},\mathsf{dc}(\mathrm{A},\mathrm{B},\mathrm{C})))}{\mathsf{dc}(\mathrm{F},\mathrm{G},\mathsf{dc}(\mathrm{E},\mathrm{D},\mathsf{dc}(\mathrm{A},\mathrm{C},\mathrm{B})))}\ \mathsf{dc}_5^{\mathsf{dc}}$$

$$\frac{\mathsf{dc}(\mathrm{H},\mathrm{I},\mathsf{dc}(\mathrm{F},\mathrm{G},\mathsf{dc}(\mathrm{D},\mathrm{E},\mathsf{dc}(\mathrm{A},\mathrm{B},\mathrm{C}))))}{\mathsf{dc}(\mathrm{H},\mathrm{I},\mathsf{dc}(\mathrm{F},\mathrm{G},\mathsf{dc}(\mathsf{dc}(\mathrm{D},\mathrm{E},\mathrm{A}),\mathsf{dc}(\mathrm{D},\mathrm{E},\mathrm{B}),\mathrm{C})))}\ \mathsf{dc}_6^{\mathsf{dc}}$$

$$\frac{\mathsf{dc}(\mathrm{H},\mathrm{I},\mathsf{dc}(\mathrm{F},\mathrm{G},\mathsf{dc}(\mathsf{dc}(\mathrm{D},\mathrm{E},\mathrm{A}),\mathsf{dc}(\mathrm{D},\mathrm{E},\mathrm{B}),\mathrm{C})))}{\mathsf{dc}(\mathrm{H},\mathrm{I},\mathsf{dc}(\mathrm{F},\mathrm{G},\mathsf{dc}(\mathrm{D},\mathrm{E},\mathsf{dc}(\mathrm{A},\mathrm{B},\mathrm{C}))))}\ \mathsf{dc}_7^{\mathsf{dc}}$$

Proof. The formally verified derivation of each rule is presented below, following what was explained in Chapter 3.

- dc_4'

```
theorem dc₄' {a b c : Prop} (h₁ : dc a b c) : dc b a c :=
  have h₂ : dc (dc b a c) (dc a b c) (dc a b c), from dc₃ h₁,
  have h₃ : dc (dc a b c) (dc b a c) (dc b a c), from dc₄ h₂,
  show dc b a c, from dc₂ h₃
```

- dc'_5

```
theorem dc₅' {a b c : Prop} (h₁ : dc a b c) : dc a c b :=
  have h₂ : dc (dc a c b) (dc a b c) (dc a b c), from dc₃ h₁,
  have h₃ : dc (dc a b c) (dc a c b) (dc a c b), from dc₅ h₂,
  show dc a c b, from dc₂ h₃
```

- dc'_6

```
theorem dc₆' {a b c d e : Prop} (h₁ : dc d e (dc a b c)) : dc (dc d e a) (dc d e b) c :=
  let f := dc d e (dc a b c), g := dc (dc d e a) (dc d e b) c in
    have h₂ : dc g f f, from dc₃ h₁,
    have h₃ : dc g f g, from dc₆ h₂,
    have h₄ : dc f g g, from dc₄' h₃,
    show g, from dc₂ h₄
```

- dc'_7

```
theorem dc₇' {a b c d e : Prop}
  (h₁ : dc (dc d e a) (dc d e b) c) : dc d e (dc a b c) :=
  let f := dc d e (dc a b c), g := dc (dc d e a) (dc d e b) c in
    have h₂ : dc f g g, from dc₃ h₁,
    have h₃ : dc f g f, from dc₇ h₂,
    have h₄ : dc g f f, from dc₄' h₃,
    show f, from dc₂ h₄
```

- dc_1^{dc}

```
theorem dc₁_dc {a b c d e : Prop} (h₁ : dc d e a) (h₂ : dc d e b) : dc d e (dc a b c) :=
  have h₂ : dc (dc d e a) (dc d e b) c, from dc₁ h₁ h₂,
  show dc d e (dc a b c), from dc₇' h₂
```

- dc_2^{dc}

```
theorem dc₂_dc {a b c d : Prop} (h₁ : dc c d (dc b a a)) : dc c d a :=
  have h₂ : dc d c (dc a b a), from dc₄ h₁,
  have h₃ : dc c d (dc a a b), from dc₅ h₂,
  have h₄ : dc (dc c d a) (dc c d a) b, from dc₆' h₃,
  have h₅ : dc b (dc c d a) (dc c d a), from dc₄' (dc₅' h₄),
  show dc c d a, from dc₂ h₅
```

- dc_3^{dc}

```
theorem dc₃_dc {a b c d : Prop} (h₁ : dc c d a) : dc c d (dc b a a) :=
  have h₂ : dc b (dc c d a) (dc c d a), from dc₃ h₁,
  have h₃ : dc (dc c d a) (dc c d a) b, from dc₅' (dc₄' h₂),
  have h₄ : dc c d (dc a a b), from dc₇' h₃,
  have h₅ : dc d c (dc a b a), from dc₅ h₄,
  show dc c d (dc b a a), from dc₄ h₅
```

- dc_4^{dc}

```
theorem dc₄_dc {a b c d e f g : Prop} (h₁ : dc f g (dc d e (dc a b c))) :
  dc f g (dc e d (dc b a c)) :=
  have h₂ : dc g f (dc e d (dc a b c)), from dc₄ h₁,
  have h₃ : dc g f (dc (dc e d a) (dc e d b) c), from dc₆ h₂,
  have h₄ : dc f g (dc (dc e d b) (dc e d a) c), from dc₄ h₃,
  show dc f g (dc e d (dc b a c)), from dc₇ h₄
```

- dc_5^{dc}

```
theorem dc₅_dc {a b c d e f g : Prop} (h₁ : dc f g (dc d e (dc a b c))) :
  dc f g (dc e d (dc a c b)) :=
  have h₂ : dc g f (dc e d (dc a b c)), from dc₄ h₁,
  have h₃ : dc (dc g f e) (dc g f d) (dc a b c), from dc₆' h₂,
  have h₄ : dc (dc g f d) (dc g f e) (dc a c b), from dc₅ h₃,
  have h₅ : dc g f (dc d e (dc a c b)), from dc₇' h₄,
  show dc f g (dc e d (dc a c b)), from dc₄ h₅
```

- dc_6^{dc}

```
theorem dc₆_dc {a b c d e f g h i : Prop} (h₁ : dc h i (dc f g (dc d e (dc a b c))))
  : dc h i (dc f g (dc (dc d e a) (dc d e b) c)) :=
```

```
have h₂ : dc (dc h i f) (dc h i g) (dc d e (dc a b c)), from dc₆' h₁,
have h₃ : dc (dc h i f) (dc h i g) (dc (dc d e a) (dc d e b) c), from dc₆ h₂,
show dc h i (dc f g (dc (dc d e a) (dc d e b) c)), from dc₇' h₃
```

- $\mathsf{dc}_7^{\mathsf{dc}}$

```
theorem dc₇_dc {a b c d e f g h i : Prop}
  (h₁ : dc h i (dc f g (dc (dc d e a) (dc d e b) c))) :
  dc h i (dc f g (dc d e (dc a b c))) :=
  have h₂ : dc (dc h i f) (dc h i g) (dc (dc d e a) (dc d e b) c), from dc₆' h₁,
  have h₃ : dc (dc h i f) (dc h i g) (dc d e (dc a b c)), from dc₇ h₂,
  show dc h i (dc f g (dc d e (dc a b c))), from dc₇' h₃
```

□

Lemma 4.11.3. *The following property holds for* $\vdash_{\mathscr{B}_{\mathsf{dc}}}$*:*

$$\begin{aligned} &\text{for all } \Gamma \cup \{\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}\} \subseteq L_{\mathsf{dc}}. \\ &\text{if } \Gamma, \mathrm{C} \vdash_{\mathscr{B}_{\mathsf{dc}}} \mathrm{D} \text{ then } \Gamma, \mathsf{dc}(\mathrm{A}, \mathrm{B}, \mathrm{C}) \vdash_{\mathscr{B}_{\mathsf{dc}}} \mathsf{dc}(\mathrm{A}, \mathrm{B}, \mathrm{D}) \end{aligned} \qquad (m_{\mathsf{dc}})$$

Proof. Let $\Gamma \cup \{\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}\} \subseteq L_{\mathsf{dc}}$ and suppose that $\Gamma, \mathrm{C} \vdash_{\mathscr{B}_{\mathsf{dc}}} \mathrm{D}$ and that this is witnessed by the n-long derivation $\mathrm{D}_1, \ldots, \mathrm{D}_n = \mathrm{D}$. Consider the property $P(i)$ given by the consecution $\Gamma, \mathsf{dc}(\mathrm{A}, \mathrm{B}, \mathrm{C}) \vdash_{\mathscr{B}_{\mathsf{dc}}} \mathsf{dc}(\mathrm{A}, \mathrm{B}, \mathrm{D}_i)$. Let us prove by induction on this derivation that $P(j)$ holds for all $1 \leq j \leq n$, and $P(n)$ will be the desired result. In the base case, there are two possibilities:

- $\mathrm{D}_1 = \mathrm{C}$, then trivially $\Gamma, \mathsf{dc}(\mathrm{A}, \mathrm{B}, \mathrm{C}) \vdash_{\mathscr{B}_{\mathsf{dc}}} \mathsf{dc}(\mathrm{A}, \mathrm{B}, \mathrm{C})$ by (R) and (M).
- $\mathrm{D}_1 \in \Gamma$, in which case we use the fact that from $\{\mathsf{dc}(\mathrm{A}, \mathrm{B}, \mathrm{C}), \mathrm{D}\}$ we get $\mathsf{dc}(\mathrm{A}, \mathrm{B}, \mathrm{D})$ by the following deduction:

(1)	$\mathsf{dc}(\mathrm{A}, \mathrm{B}, \mathrm{C})$	Assumption
(2)	D	Assumption
(3)	$\mathsf{dc}(\mathsf{dc}(\mathrm{A}, \mathrm{B}, \mathrm{C}), \mathrm{D}, \mathsf{dc}(\mathrm{A}, \mathrm{B}, \mathrm{D}))$	$1, 2$ dc_1
(4)	$\mathsf{dc}(\mathsf{dc}(\mathrm{A}, \mathrm{B}, \mathrm{C}), \mathsf{dc}(\mathrm{A}, \mathrm{B}, \mathrm{D}), \mathrm{D})$	3 dc_5'
(5)	$\mathsf{dc}(\mathrm{A}, \mathrm{B}, \mathsf{dc}(\mathrm{C}, \mathrm{D}, \mathrm{D}))$	4 dc_7'
(6)	$\mathsf{dc}(\mathrm{A}, \mathrm{B}, \mathrm{D})$	5 $\mathsf{dc}_2^{\mathsf{dc}}$

For the inductive step, suppose that $P(j)$ holds for all $1 \leq j < k$, where $k > 1$. Then, there are three cases to consider regarding D_k: two of them are the same as in the base case, while the other considers D_k as a result from one of the m-ary rules of $\mathscr{B}_{\mathsf{dc}}$, say $\mathsf{dc_i}$, by the instance $\langle \mathrm{D}_{k_1}, \ldots, \mathrm{D}_{k_m}, \mathrm{B}_k \rangle$, where $k_l < k$, for all $1 \leq l \leq m$. By the inductive hypothesis, the formulas $\mathsf{dc}(\mathrm{A}, \mathrm{B}, \mathrm{D}_{k_1}), \ldots, \mathsf{dc}(\mathrm{A}, \mathrm{B}, \mathrm{D}_{k_m})$ are all derivable from $\Gamma \cup \{\mathsf{dc}(\mathrm{A}, \mathrm{B}, \mathrm{C})\}$. Then simply apply to such formulas the dc-lifted version of rule $\mathsf{dc_i}$, namely $\mathsf{dc_i^{dc}}$, proved to be derivable in Lemma 4.11.2, to obtain the desired $\mathsf{dc}(\mathrm{A}, \mathrm{B}, \mathrm{D}_k)$ from $\Gamma \cup \{\mathsf{dc}(\mathrm{A}, \mathrm{B}, \mathrm{C})\}$. □

Theorem 4.11.4. *The following property holds for* $\vdash_{\mathscr{B}_{\mathsf{dc}}}$*:*

$$\begin{array}{c} \text{for all } \Gamma \cup \{\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}\} \subseteq L_{\mathsf{dc}} \\ \text{if } \Gamma, \mathrm{B} \vdash_{\mathscr{B}_{\mathsf{dc}}} \mathrm{D} \text{ and } \Gamma, \mathrm{C} \vdash_{\mathscr{B}_{\mathsf{dc}}} \mathrm{D} \text{ then } \Gamma, \mathsf{dc}(\mathrm{A}, \mathrm{B}, \mathrm{C}) \vdash_{\mathscr{B}_{\mathsf{dc}}} \mathrm{D} \end{array} \qquad (\delta_{\mathsf{dc}})$$

Proof. Let $\Gamma \cup \{\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}\} \subseteq L_{\mathsf{dc}}$ and suppose that $\Gamma, \mathrm{B} \vdash_{\mathscr{B}_{\mathsf{dc}}} \mathrm{D}$ and $\Gamma, \mathrm{C} \vdash_{\mathscr{B}_{\mathsf{dc}}} \mathrm{D}$. By m_{dc}, we have (a): $\Gamma, \mathsf{dc}(\mathrm{A}, \mathrm{D}, \mathrm{B}) \vdash_{\mathscr{B}_{\mathsf{dc}}} \mathsf{dc}(\mathrm{A}, \mathrm{D}, \mathrm{D})$ and (b): $\Gamma, \mathsf{dc}(\mathrm{A}, \mathrm{B}, \mathrm{C}) \vdash_{\mathscr{B}_{\mathsf{dc}}} \mathsf{dc}(\mathrm{A}, \mathrm{B}, \mathrm{D})$. From (a), by $\mathsf{dc_2}$, we get (a'): $\Gamma, \mathsf{dc}(\mathrm{A}, \mathrm{D}, \mathrm{B}) \vdash_{\mathscr{B}_{\mathsf{dc}}} \mathrm{D}$. From (b), by dc'_5, we get (b'): $\Gamma, \mathsf{dc}(\mathrm{A}, \mathrm{B}, \mathrm{C}) \vdash_{\mathscr{B}_{\mathsf{dc}}} \mathsf{dc}(\mathrm{A}, \mathrm{D}, \mathrm{B})$. Finally, by (T) applied to (a') and (b'), we get the desired result $\Gamma, \mathsf{dc}(\mathrm{A}, \mathrm{B}, \mathrm{C}) \vdash_{\mathscr{B}_{\mathsf{dc}}} \mathrm{D}$. □

Theorem 4.11.5. *The calculus* $\mathscr{B}_{\mathsf{dc}}$ *is complete with respect to the matrix* $\mathbb{2}_{\mathsf{dc}}$*.*

Proof. According to the procedure presented in Section 2.7, let $\Gamma \cup \{\mathrm{Z}\} \subseteq L_{\mathsf{dc}}$ and take the Z-maximal theory $\Gamma^+ \supseteq \Gamma$ via the Lindenbaum-Asser Lemma. The inspection of the truth-table for dc in $\mathbb{2}_{\mathsf{dc}}$ reveals the completeness property that needs to be proved:

$$\mathsf{dc}(\mathrm{A}, \mathrm{B}, \mathrm{C}) \in \Gamma^+ \text{ iff } \mathrm{A}, \mathrm{B} \in \Gamma^+ \text{ or } \mathrm{A}, \mathrm{C} \in \Gamma^+ \text{ or } \mathrm{B}, \mathrm{C} \in \Gamma^+. \qquad (\mathsf{dc})$$

From the left to the right, suppose that (a): $\Gamma^+ \vdash_{\mathscr{B}_{\mathsf{dc}}} \mathsf{dc}(\mathrm{A}, \mathrm{B}, \mathrm{C})$. Proceed by contradiction: suppose first that $\mathrm{A}, \mathrm{B} \notin \Gamma^+$, thus $\Gamma^+, \mathrm{A} \vdash_{\mathscr{B}_{\mathsf{dc}}} \mathrm{Z}$ and $\Gamma^+, \mathrm{B} \vdash_{\mathscr{B}_{\mathsf{dc}}} \mathrm{Z}$. Then, by δ_{dc}, we get $\Gamma^+, \mathsf{dc}(\mathrm{A}, \mathrm{B}, \mathrm{C}) \vdash_{\mathscr{B}_{\mathsf{dc}}} \mathrm{Z}$ and, by (T) applied to the latter and (a), we get $\Gamma^+ \vdash_{\mathscr{B}_{\mathsf{dc}}} \mathrm{Z}$, a contradiction. The other two cases are analogous with the help of the permuting rules dc'_4 and dc'_5. Next, from the right to the left, we need to consider the three cases $\mathrm{A}, \mathrm{B} \in \Gamma^+$, $\mathrm{A}, \mathrm{C} \in \Gamma^+$ and $\mathrm{B}, \mathrm{C} \in \Gamma^+$. Suppose first that $\mathrm{A}, \mathrm{B} \in \Gamma^+$. Then, $\Gamma^+ \vdash_{\mathscr{B}_{\mathsf{dc}}} \mathrm{A}$ and $\Gamma^+ \vdash_{\mathscr{B}_{\mathsf{dc}}} \mathrm{B}$. By rule $\mathsf{dc_1}$, $\Gamma^+ \vdash_{\mathscr{B}_{\mathsf{dc}}} \mathsf{dc}(\mathrm{A}, \mathrm{B}, \mathrm{C})$. Suppose now that $\mathrm{A}, \mathrm{C} \in \Gamma^+$. Then, $\Gamma^+ \vdash_{\mathscr{B}_{\mathsf{dc}}} \mathrm{A}$ and $\Gamma^+ \vdash_{\mathscr{B}_{\mathsf{dc}}} \mathrm{C}$. By rule $\mathsf{dc_1}$, $\Gamma^+ \vdash_{\mathscr{B}_{\mathsf{dc}}} \mathsf{dc}(\mathrm{A}, \mathrm{C}, \mathrm{B})$, and, by dc'_5, we get $\Gamma^+ \vdash_{\mathscr{B}_{\mathsf{dc}}} \mathsf{dc}(\mathrm{A}, \mathrm{B}, \mathrm{C})$. The third case is similar, the only difference being the application of dc'_4 and dc'_5, instead of only the former. □

Remark 4.11.1. Notice that a sufficient condition for the preservation of the property m_{dc}, and thus the completeness property (dc), in any expansion of the calculus $\mathscr{B}_{\mathsf{dc}}$ by non-nullary rules is that, for any of the new rules, say r, its dc-lifted version r^{dc} is derivable in the expanded calculus.

4.12 $\mathcal{B}_{\mathsf{pt,dc}}$

The proposed calculus for the fragment $\mathcal{B}_{\mathsf{pt,dc}}$ is one of the most complex calculi present in this work. Rautenberg, in [11], just indicated that adding rules dcpt_1 and dcpt_3 (see below) to the merging of $\mathscr{B}_{\mathsf{pt}}$ and $\mathscr{B}_{\mathsf{dc}}$ is the path for finding the desired axiomatization, without presenting the full calculus or any details about its completeness with respect to $\mathcal{2}_{\mathsf{pt,dc}}$. Here, we have found that adding the converses of the mentioned rules (dcpt_2 and dcpt_4 below, respectively), together with the dc-lifted versions of dcpt_3 and dcpt_4 and the pt-lifted versions of dcpt_5 and dcpt_6 are enough to complete the axiomatization suggested by Rautenberg. See, in the sequel, the resulting calculus followed by the proof of soundness with respect to $\mathcal{2}_{\mathsf{pt,dc}}$.

Hilbert Calculus 30. $\mathscr{B}_{\mathsf{pt,dc}}$

$\mathscr{B}_{\mathsf{pt}}$ $\quad$ $\mathscr{B}_{\mathsf{dc}}$

$$\frac{\mathsf{dc}(A, B, \mathsf{pt}(C, D, E))}{\mathsf{pt}(\mathsf{dc}(A, B, C), \mathsf{dc}(A, B, D), \mathsf{dc}(A, B, E))}\ \mathsf{dcpt}_1$$

$$\frac{\mathsf{pt}(\mathsf{dc}(A, B, C), \mathsf{dc}(A, B, D), \mathsf{dc}(A, B, E))}{\mathsf{dc}(A, B, \mathsf{pt}(C, D, E))}\ \mathsf{dcpt}_2$$

$$\frac{\mathsf{pt}(A, B, \mathsf{dc}(C, D, E))}{\mathsf{dc}(\mathsf{pt}(A, B, C), \mathsf{pt}(A, B, D), \mathsf{pt}(A, B, E))}\ \mathsf{dcpt}_3$$

$$\frac{\mathsf{dc}(\mathsf{pt}(A, B, C), \mathsf{pt}(A, B, D), \mathsf{pt}(A, B, E))}{\mathsf{pt}(A, B, \mathsf{dc}(C, D, E))}\ \mathsf{dcpt}_4$$

$$\frac{\mathsf{dc}(F, G, \mathsf{pt}(A, B, \mathsf{dc}(C, D, E)))}{\mathsf{dc}(F, G, \mathsf{dc}(\mathsf{pt}(A, B, C), \mathsf{pt}(A, B, D), \mathsf{pt}(A, B, E)))}\ \mathsf{dcpt}_5$$

$$\frac{\mathsf{dc}(F, G, \mathsf{dc}(\mathsf{pt}(A, B, C), \mathsf{pt}(A, B, D), \mathsf{pt}(A, B, E)))}{\mathsf{dc}(F, G, \mathsf{pt}(A, B, \mathsf{dc}(C, D, E)))}\ \mathsf{dcpt}_6$$

$$\frac{\mathsf{pt}(F, G, \mathsf{dc}(A, B, \mathsf{pt}(C, D, E)))}{\mathsf{pt}(F, G, \mathsf{pt}(\mathsf{dc}(A, B, C), \mathsf{dc}(A, B, D), \mathsf{dc}(A, B, E)))}\ \mathsf{dcpt}_7$$

$$\frac{\mathsf{pt}(\mathrm{F}, \mathrm{G}, \mathsf{pt}(\mathsf{dc}(\mathrm{A}, \mathrm{B}, \mathrm{C}), \mathsf{dc}(\mathrm{A}, \mathrm{B}, \mathrm{D}), \mathsf{dc}(\mathrm{A}, \mathrm{B}, \mathrm{E})))}{\mathsf{pt}(\mathrm{F}, \mathrm{G}, \mathsf{dc}(\mathrm{A}, \mathrm{B}, \mathsf{pt}(\mathrm{C}, \mathrm{D}, \mathrm{E})))}\ \mathsf{dcpt}_8$$

Theorem 4.12.1. *The calculus $\mathscr{B}_{\mathsf{pt,dc}}$ is sound with respect to the matrix $\mathcal{2}_{\mathsf{pt,dc}}$.*

Proof. Let v be a $\mathcal{2}_{\mathsf{pt,dc}}$-valuation. For dcpt_1, consider the cases in which v assigns 1 to $\mathsf{dc}(\mathrm{A}, \mathrm{B}, \mathsf{pt}(\mathrm{C}, \mathrm{D}, \mathrm{E}))$:

- $v(\mathrm{A}) = 1$, $v(\mathrm{B}) = 1$: these assignments to A and B cause $v(\mathsf{dc}(\mathrm{A}, \mathrm{B}, \cdot)) = 1$, thus v assigns 1 to $\mathsf{pt}(\mathsf{dc}(\mathrm{A}, \mathrm{B}, \mathrm{C}), \mathsf{dc}(\mathrm{A}, \mathrm{B}, \mathrm{D}), \mathsf{dc}(\mathrm{A}, \mathrm{B}, \mathrm{E}))$.
- $v(\mathrm{A}) = 1$, $v(\mathrm{B}) = 0$ and $v(\mathsf{pt}(\mathrm{C}, \mathrm{D}, \mathrm{E})) = 1$: the assignment $v(\mathsf{pt}(\mathrm{C}, \mathrm{D}, \mathrm{E})) = 1$ means that all of the formulas C, D and E are assigned the value 1 or only one of them. In the first case, we will have $v(\mathsf{dc}(\mathrm{A}, \mathrm{B}, \mathrm{C})) = 1$, $v(\mathsf{dc}(\mathrm{A}, \mathrm{B}, \mathrm{D})) = 1$ and $v(\mathsf{dc}(\mathrm{A}, \mathrm{B}, \mathrm{E})) = 1$, so the conclusion gets the value 1 under v. In the other case, only one of those formulas are assigned to 1 under v, what also makes $v(\mathsf{pt}(\mathsf{dc}(\mathrm{A}, \mathrm{B}, \mathrm{C}), \mathsf{dc}(\mathrm{A}, \mathrm{B}, \mathrm{D}), \mathsf{dc}(\mathrm{A}, \mathrm{B}, \mathrm{E}))) = 1$.
- $v(\mathrm{A}) = 0$, $v(\mathrm{B}) = 1$ and $v(\mathsf{pt}(\mathrm{C}, \mathrm{D}, \mathrm{E})) = 1$: similar to the previous case.

For rule dcpt_2, suppose that v assigns 0 to $\mathsf{dc}(\mathrm{A}, \mathrm{B}, \mathsf{pt}(\mathrm{C}, \mathrm{D}, \mathrm{E}))$ and consider the cases that leads to this. A reasoning dual to the one used previously shows that the premiss of dcpt_2 is always evaluated to 0 under this condition. For rule dcpt_3, suppose that v assigns 1 to $\mathsf{pt}(\mathrm{A}, \mathrm{B}, \mathsf{dc}(\mathrm{C}, \mathrm{D}, \mathrm{E}))$. Then we have the following cases to consider:

- $v(\mathrm{A}) = 1$, $v(\mathrm{B}) = 0$ and $v(\mathsf{dc}(\mathrm{C}, \mathrm{D}, \mathrm{E})) = 0$: from $v(\mathsf{dc}(\mathrm{C}, \mathrm{D}, \mathrm{E})) = 0$ we have the cases
  - $v(\mathrm{C}) = 0$, $v(\mathrm{D}) = 0$ and $v(\mathrm{E}) = 0$: since $v(\mathrm{A}) = 1$ and $v(\mathrm{B}) = 0$, we have $v(\mathsf{pt}(\mathrm{A}, \mathrm{B}, \mathrm{C})) = 1$, $v(\mathsf{pt}(\mathrm{A}, \mathrm{B}, \mathrm{D})) = 1$, $v(\mathsf{pt}(\mathrm{A}, \mathrm{B}, \mathrm{E})) = 1$, so v also assigns 1 to the conclusion.
  - $v(\mathrm{C}) = 1$, $v(\mathrm{D}) = 0$ and $v(\mathrm{E}) = 0$: since $v(\mathrm{A}) = 1$ and $v(\mathrm{B}) = 0$, we have $v(\mathsf{pt}(\mathrm{A}, \mathrm{B}, \mathrm{C})) = 1$, while $v(\mathsf{pt}(\mathrm{A}, \mathrm{B}, \mathrm{D})) = 0$, $v(\mathsf{pt}(\mathrm{A}, \mathrm{B}, \mathrm{E})) = 0$, so v also assigns 1 to the conclusion.
  - $v(\mathrm{C}) = 0$, $v(\mathrm{D}) = 1$ and $v(\mathrm{E}) = 0$, or $v(\mathrm{C}) = 0$, $v(\mathrm{D}) = 0$ and $v(\mathrm{E}) = 1$: similar to the previous case.
- $v(\mathrm{A}) = 0$, $v(\mathrm{B}) = 1$ and $v(\mathsf{dc}(\mathrm{C}, \mathrm{D}, \mathrm{E})) = 0$: analogous to the case above.

- $v(\mathrm{A}) = 0$, $v(\mathrm{B}) = 0$ and $v(\mathsf{dc}(\mathrm{C}, \mathrm{D}, \mathrm{E})) = 1$: in view of $v(\mathsf{dc}(\mathrm{C}, \mathrm{D}, \mathrm{E})) = 1$, consider the cases
  - $v(\mathrm{C}) = 1$, $v(\mathrm{D}) = 1$ and $v(\mathrm{E}) = 1$: since $v(\mathrm{A}) = v(\mathrm{B}) = 0$, we have $v(\mathsf{pt}(\mathrm{A}, \mathrm{B}, \mathrm{C})) = 1$, $v(\mathsf{pt}(\mathrm{A}, \mathrm{B}, \mathrm{D})) = 1$, $v(\mathsf{pt}(\mathrm{A}, \mathrm{B}, \mathrm{E})) = 1$, so the conclusion gets the value 1 under v.
  - $v(\mathrm{C}) = 1$, $v(\mathrm{D}) = 1$ and $v(\mathrm{E}) = 0$: since $v(\mathrm{A}) = v(\mathrm{B}) = 0$, we have $v(\mathsf{pt}(\mathrm{A}, \mathrm{B}, \mathrm{C})) = 1$, $v(\mathsf{pt}(\mathrm{A}, \mathrm{B}, \mathrm{D})) = 1$, $v(\mathsf{pt}(\mathrm{A}, \mathrm{B}, \mathrm{E})) = 0$, so the conclusion gets the value 1 under v.
  - $v(\mathrm{C}) = 0$, $v(\mathrm{D}) = 1$ and $v(\mathrm{E}) = 1$, or $v(\mathrm{C}) = 1$, $v(\mathrm{D}) = 0$ and $v(\mathrm{E}) = 1$: analogous to the previous case.
- $v(\mathrm{A}) = 1$, $v(\mathrm{B}) = 1$ and $v(\mathsf{dc}(\mathrm{C}, \mathrm{D}, \mathrm{E})) = 1$: analogous to the case above.

For dcpt_4, assume that v assigns 0 to $\mathsf{pt}(\mathrm{A}, \mathrm{B}, \mathsf{dc}(\mathrm{C}, \mathrm{D}, \mathrm{E}))$ and proceed similarly to the previous proof, but considering the four cases that emerge from this assumption, in order to show that the premiss is always assigned the value 0 under v. The soundness of the remaining rules follows from the previous ones and the fact that if we have $v(\mathrm{Q}) = v(\mathrm{Q}')$, then $v(\#(\mathrm{A}, \mathrm{B}, \mathrm{Q})) = v(\#(\mathrm{A}, \mathrm{B}, \mathrm{Q}'))$, for some ternary connective $\#$. □

The primitive rules of the proposed calculus allow us to derive their pt- and dc-lifted versions, which are what we need to prove the completeness of $\mathscr{B}_{\mathsf{pt},\mathsf{dc}}$ with respect to $\mathbb{2}_{\mathsf{pt},\mathsf{dc}}$. The lemma below presents such derivations together with some auxiliar rules.

Lemma 4.12.2. *The following rules are derivable in* $\mathscr{B}_{\mathsf{dc},\mathsf{pt}}$*:*

$$\frac{\mathsf{dc}(\mathrm{D}, \mathrm{E}, \mathrm{A}) \quad \mathsf{dc}(\mathrm{D}, \mathrm{E}, \mathrm{B}) \quad \mathsf{dc}(\mathrm{D}, \mathrm{E}, \mathrm{C})}{\mathsf{dc}(\mathrm{D}, \mathrm{E}, \mathsf{pt}(\mathrm{A}, \mathrm{B}, \mathrm{C}))}\ \mathsf{pt}_1^{\mathsf{dc}}$$

$$\frac{\mathsf{dc}(\mathrm{D}, \mathrm{E}, \mathsf{pt}(\mathrm{A}, \mathrm{B}, \mathrm{C}))}{\mathsf{dc}(\mathrm{D}, \mathrm{E}, \mathsf{pt}(\mathrm{B}, \mathrm{A}, \mathrm{C}))}\ \mathsf{pt}_2^{\mathsf{dc}}$$

$$\frac{\mathsf{dc}(\mathrm{D}, \mathrm{E}, \mathsf{pt}(\mathrm{A}, \mathrm{B}, \mathrm{C}))}{\mathsf{dc}(\mathrm{D}, \mathrm{E}, \mathsf{pt}(\mathrm{A}, \mathrm{C}, \mathrm{B}))}\ \mathsf{pt}_3^{\mathsf{dc}}$$

$$\frac{\mathsf{dc}(\mathrm{C}, \mathrm{D}, \mathrm{A})}{\mathsf{dc}(\mathrm{C}, \mathrm{D}, \mathsf{pt}(\mathrm{A}, \mathrm{B}, \mathrm{B}))}\ \mathsf{pt}_4^{\mathsf{dc}}$$

$$\frac{\mathsf{dc}(\mathrm{C}, \mathrm{D}, \mathsf{pt}(\mathrm{A}, \mathrm{B}, \mathrm{B}))}{\mathsf{dc}(\mathrm{C}, \mathrm{D}, \mathrm{A})}\ \mathsf{pt}_5^{\mathsf{dc}}$$

$$\frac{\mathsf{dc}(\mathrm{F}, \mathrm{G}, \mathsf{pt}(\mathrm{A}, \mathrm{B}, \mathsf{pt}(\mathrm{C}, \mathrm{D}, \mathrm{E})))}{\mathsf{dc}(\mathrm{F}, \mathrm{G}, \mathsf{pt}(\mathsf{pt}(\mathrm{A}, \mathrm{B}, \mathrm{C}), \mathrm{D}, \mathrm{E}))}\ \mathsf{pt}_6^{\mathsf{dc}}$$

$$\frac{\mathsf{dc}(F, G, \mathsf{pt}(\mathsf{pt}(A, B, C), D, E))}{\mathsf{dc}(F, G, \mathsf{pt}(A, B, \mathsf{pt}(C, D, E)))}\ \mathsf{pt}_7^{\mathsf{dc}}$$

$$\frac{\mathsf{pt}(D, E, A) \quad \mathsf{pt}(D, E, B)}{\mathsf{pt}(D, E, \mathsf{dc}(A, B, C))}\ \mathsf{dc}_1^{\mathsf{pt}}$$

$$\frac{\mathsf{pt}(C, D, \mathsf{dc}(B, A, A))}{\mathsf{pt}(C, D, A)}\ \mathsf{dc}_2^{\mathsf{pt}}$$

$$\frac{\mathsf{pt}(C, D, A)}{\mathsf{pt}(C, D, \mathsf{dc}(B, A, A))}\ \mathsf{dc}_3^{\mathsf{pt}}$$

$$\frac{\mathsf{pt}(F, G, \mathsf{dc}(D, E, \mathsf{dc}(A, B, C)))}{\mathsf{pt}(F, G, \mathsf{dc}(E, D, \mathsf{dc}(B, A, C)))}\ \mathsf{dc}_4^{\mathsf{pt}}$$

$$\frac{\mathsf{pt}(F, G, \mathsf{dc}(D, E, \mathsf{dc}(A, B, C)))}{\mathsf{pt}(F, G, \mathsf{dc}(E, D, \mathsf{dc}(A, C, B)))}\ \mathsf{dc}_5^{\mathsf{pt}}$$

$$\frac{\mathsf{dc}(H, I, \mathsf{dc}(F, G, \mathsf{pt}(A, B, \mathsf{dc}(C, D, E))))}{\mathsf{dc}(H, I, \mathsf{dc}(F, G, \mathsf{dc}(\mathsf{pt}(A, B, C), \mathsf{pt}(A, B, D), \mathsf{pt}(A, B, E)))))}\ \mathsf{dcpt}_5^{\mathsf{dc}}$$

$$\frac{\mathsf{dc}(H, I, \mathsf{dc}(F, G, \mathsf{dc}(\mathsf{pt}(A, B, C), \mathsf{pt}(A, B, D), \mathsf{pt}(A, B, E)))))}{\mathsf{dc}(H, I, \mathsf{dc}(F, G, \mathsf{pt}(A, B, \mathsf{dc}(C, D, E))))}\ \mathsf{dcpt}_6^{\mathsf{dc}}$$

$$\frac{\mathsf{pt}(H, I, \mathsf{dc}(F, G, \mathsf{dc}(D, E, \mathsf{dc}(A, B, C))))}{\mathsf{pt}(H, I, \mathsf{dc}(F, G, \mathsf{dc}(\mathsf{dc}(D, E, A), \mathsf{dc}(D, E, B), C)))}\ \mathsf{dc}_6^{\mathsf{pt}}$$

$$\frac{\mathsf{pt}(H, I, \mathsf{dc}(F, G, \mathsf{dc}(\mathsf{dc}(D, E, A), \mathsf{dc}(D, E, B), C)))}{\mathsf{pt}(H, I, \mathsf{dc}(F, G, \mathsf{dc}(D, E, \mathsf{dc}(A, B, C))))}\ \mathsf{dc}_7^{\mathsf{pt}}$$

$$\frac{\mathsf{pt}(D, E, \mathsf{pt}(A, B, C))}{\mathsf{pt}(D, E, \mathsf{dc}(B, A, C))}\ \mathsf{dc}'^{\mathsf{pt}}_4$$

$$\frac{\mathsf{pt}(D, E, \mathsf{dc}(A, B, C))}{\mathsf{pt}(D, E, \mathsf{dc}(A, C, B))}\ \mathsf{dc}'^{\mathsf{pt}}_5$$

$$\frac{\mathsf{pt}(F, G, \mathsf{dc}(D, E, \mathsf{dc}(A, B, C)))}{\mathsf{pt}(F, G, \mathsf{dc}(\mathsf{dc}(D, E, A), \mathsf{dc}(D, E, B), C))}\ \mathsf{dc}'^{\mathsf{pt}}_6$$

$$\frac{\mathsf{pt}(F, G, \mathsf{dc}(\mathsf{dc}(D, E, A), \mathsf{dc}(D, E, B), C))}{\mathsf{pt}(F, G, \mathsf{dc}(D, E, \mathsf{dc}(A, B, C)))}\ \mathsf{dc}'^{\mathsf{pt}}_7$$

$$\frac{\mathsf{pt}(F, G, \mathsf{dc}(A, B, \mathsf{pt}(C, D, E)))}{\mathsf{pt}(F, G, \mathsf{pt}(\mathsf{dc}(A, B, C), \mathsf{dc}(A, B, D), \mathsf{dc}(A, B, E)))}\ \mathsf{dcpt}_1^{\mathsf{pt}}$$

$$\frac{\mathsf{pt}(F, G, \mathsf{pt}(\mathsf{dc}(A, B, C), \mathsf{dc}(A, B, D), \mathsf{dc}(A, B, E)))}{\mathsf{pt}(F, G, \mathsf{dc}(A, B, \mathsf{pt}(C, D, E)))}\ \mathsf{dcpt}_2^{\mathsf{pt}}$$

$$\frac{\mathsf{pt}(F, G, \mathsf{pt}(A, B, \mathsf{dc}(C, D, E)))}{\mathsf{pt}(F, G, \mathsf{dc}(\mathsf{pt}(A, B, C), \mathsf{pt}(A, B, D), \mathsf{pt}(A, B, E)))}\ \mathsf{dcpt}_3^{\mathsf{pt}}$$

$$\frac{\mathsf{pt}(F, G, \mathsf{dc}(\mathsf{pt}(A, B, C), \mathsf{pt}(A, B, D), \mathsf{pt}(A, B, E)))}{\mathsf{pt}(F, G, \mathsf{pt}(A, B, \mathsf{dc}(C, D, E)))}\ \mathsf{dcpt}_4^{\mathsf{pt}}$$

$$\frac{\mathsf{pt}(H, I, \mathsf{dc}(F, G, \mathsf{pt}(A, B, \mathsf{dc}(C, D, E))))}{\mathsf{pt}(H, I, \mathsf{dc}(F, G, \mathsf{dc}(\mathsf{pt}(A, B, C), \mathsf{pt}(A, B, D), \mathsf{pt}(A, B, E)))))}\ \mathsf{dcpt}_5^{\mathsf{pt}}$$

$$\frac{\mathsf{pt}(H, I, \mathsf{dc}(F, G, \mathsf{dc}(\mathsf{pt}(A, B, C), \mathsf{pt}(A, B, D), \mathsf{pt}(A, B, E))))}{\mathsf{pt}(H, I, \mathsf{dc}(F, G, \mathsf{pt}(A, B, \mathsf{dc}(C, D, E))))}\ \mathsf{dcpt}_6^{\mathsf{pt}}$$

$$\frac{\mathsf{pt}(H, I, \mathsf{pt}(F, G, \mathsf{dc}(A, B, \mathsf{pt}(C, D, E))))}{\mathsf{pt}(H, I, \mathsf{pt}(F, G, \mathsf{pt}(\mathsf{dc}(A, B, C), \mathsf{dc}(A, B, D), \mathsf{dc}(A, B, E))))}\ \mathsf{dcpt}_7^{\mathsf{pt}}$$

$$\frac{\mathsf{pt}(H, I, \mathsf{pt}(F, G, \mathsf{pt}(\mathsf{dc}(A, B, C), \mathsf{dc}(A, B, D), \mathsf{dc}(A, B, E))))}{\mathsf{pt}(H, I, \mathsf{pt}(F, G, \mathsf{dc}(A, B, \mathsf{pt}(C, D, E))))}\ \mathsf{dcpt}_8^{\mathsf{pt}}$$

$$\frac{\mathsf{dc}(F, G, \mathsf{dc}(A, B, \mathsf{pt}(C, D, E)))}{\mathsf{dc}(F, G, \mathsf{pt}(\mathsf{dc}(A, B, C), \mathsf{dc}(A, B, D), \mathsf{dc}(A, B, E)))}\ \mathsf{dcpt}_1^{\mathsf{dc}}$$

$$\frac{\mathsf{dc}(F, G, \mathsf{pt}(\mathsf{dc}(A, B, C), \mathsf{dc}(A, B, D), \mathsf{dc}(A, B, E)))}{\mathsf{dc}(F, G, \mathsf{dc}(A, B, \mathsf{pt}(C, D, E)))}\ \mathsf{dcpt}_2^{\mathsf{dc}}$$

$$\frac{\mathsf{dc}(F, G, \mathsf{pt}(A, B, \mathsf{dc}(C, D, E)))}{\mathsf{dc}(F, G, \mathsf{dc}(\mathsf{pt}(A, B, C), \mathsf{pt}(A, B, D), \mathsf{pt}(A, B, E)))}\ \mathsf{dcpt}_3^{\mathsf{dc}}$$

$$\frac{\mathsf{dc}(F, G, \mathsf{dc}(\mathsf{pt}(A, B, C), \mathsf{pt}(A, B, D), \mathsf{pt}(A, B, E)))}{\mathsf{dc}(F, G, \mathsf{pt}(A, B, \mathsf{dc}(C, D, E)))}\ \mathsf{dcpt}_4^{\mathsf{dc}}$$

$$\frac{\mathsf{dc}(H, I, \mathsf{pt}(F, G, \mathsf{dc}(A, B, \mathsf{pt}(C, D, E))))}{\mathsf{dc}(H, I, \mathsf{pt}(F, G, \mathsf{pt}(\mathsf{dc}(A, B, C), \mathsf{dc}(A, B, D), \mathsf{dc}(A, B, E))))}\ \mathsf{dcpt}_7^{\mathsf{dc}}$$

$$\frac{\mathsf{dc}(H, I, \mathsf{pt}(F, G, \mathsf{pt}(\mathsf{dc}(A, B, C), \mathsf{dc}(A, B, D), \mathsf{dc}(A, B, E))))}{\mathsf{dc}(H, I, \mathsf{pt}(F, G, \mathsf{dc}(A, B, \mathsf{pt}(C, D, E))))}\ \mathsf{dcpt}_8^{\mathsf{dc}}$$

Proof. The formally verified derivation of each rule is presented below, following what was explained in Chapter 3.

- $\mathsf{pt}_1^{\mathsf{dc}}$

```
theorem pt₁_dc {a b c d e : Prop} (h₁ : dc d e a) (h₂ : dc d e b) (h₃ : dc d e c)
    : dc d e (pt a b c) := dcpt₂ (pt.pt₁ h₁ h₂ h₃)
```

- $\mathsf{pt}_2^{\mathsf{dc}}$

```
theorem pt₂_dc {a b c d e : Prop} (h₁ : dc d e (pt a b c)) : dc d e (pt b a c) :=
  dcpt₂ (pt.pt₂ (dcpt₁ h₁))
```

- $\mathsf{pt}_3^{\mathsf{dc}}$

```
theorem pt₃_dc {a b c d e : Prop} (h₁ : dc d e (pt a b c)) : dc d e (pt a c b) :=
  dcpt₂ (pt.pt₃ (dcpt₁ h₁))
```

- $\mathsf{pt}_4^{\mathsf{dc}}$

```
theorem pt₄_dc {a b c d : Prop} (h₁ : dc c d a) : dc c d (pt a b b) :=
  dcpt₂ (pt.pt₄ h₁)
```

- $\mathsf{pt}_5^{\mathsf{dc}}$

```
theorem pt₅_dc {a b c d : Prop} (h₁ : dc c d (pt a b b)) : dc c d a :=
  pt.pt₅ (dcpt₁ h₁)
```

- $\mathsf{pt}_6^{\mathsf{dc}}$

```
theorem pt₆_dc {a b c d e f g : Prop} (h₁ : dc f g (pt a b (pt c d e))) :
  dc f g (pt (pt a b c) d e) :=
  have h₂ : pt (dc f g a) (dc f g b) (dc f g (pt c d e)),
    from dcpt₁ h₁,
  have h₃ : pt (dc f g a) (dc f g b) (pt (dc f g c) (dc f g d) (dc f g e)),
    from dcpt₇ h₂,
  have h₄ : pt (pt (dc f g a) (dc f g b) (dc f g c)) (dc f g d) (dc f g e),
    from pt.pt₆ h₃,
  have h₅ : pt (dc f g d) (dc f g e) (pt (dc f g a) (dc f g b) (dc f g c)),
    from pt.pt₃ (pt.pt₂ h₄),
  have h₆ : pt (dc f g d) (dc f g e) (dc f g (pt a b c)),
    from dcpt₈ h₅,
  have h₇ : pt (dc f g (pt a b c)) (dc f g d) (dc f g e),
    from pt.pt₂ (pt.pt₃ h₆),
  show dc f g (pt (pt a b c) d e), from dcpt₂ h₇
```

- $\mathsf{pt}_7^{\mathsf{dc}}$

```
theorem pt₇_dc {a b c d e f g : Prop}
  (h₁ : dc f g (pt (pt a b c) d e)) :
  dc f g (pt a b (pt c d e)) :=
  have h₂ : dc f g (pt d (pt a b c) e), from pt₂_dc h₁,
  have h₃ : dc f g (pt d e (pt a b c)), from pt₃_dc h₂,
  have h₄ : dc f g (pt (pt d e a) b c), from pt₆_dc h₃,
  have h₅ : dc f g (pt b (pt d e a) c), from pt₂_dc h₄,
  have h₆ : dc f g (pt b c (pt d e a)), from pt₃_dc h₅,
  have h₇ : dc f g (pt (pt b c d) e a), from pt₆_dc h₆,
  have h₈ : dc f g (pt e (pt b c d) a), from pt₂_dc h₇,
  have h₉ : dc f g (pt e a (pt b c d)), from pt₃_dc h₈,
  have h₁₀ : dc f g (pt (pt e a b) c d), from pt₆_dc h₉,
```

```
    have h₁₁ : dc f g (pt c (pt e a b) d), from pt₂_dc h₁₀,
    have h₁₂ : dc f g (pt c d (pt e a b)), from pt₃_dc h₁₁,
    have h₁₃ : dc f g (pt (pt c d e) a b), from pt₆_dc h₁₂,
    have h₁₄ : dc f g (pt a (pt c d e) b), from pt₂_dc h₁₃,
    show dc f g (pt a b (pt c d e)), from pt₃_dc h₁₄
```

- dc_1^{pt}

```
theorem dc₁_pt {a b c d e : Prop}  (h₁ : pt d e a) (h₂ : pt d e b) : pt d e (dc a b c) :=
    have h₃ : dc (pt d e a) (pt d e b) (pt d e c), from dc.dc₁ h₁ h₂,
    show pt d e (dc a b c), from dcpt₄ h₃
```

- dc_2^{pt}

```
theorem dc₂_pt {a b c d : Prop} (h₁ : pt c d (dc b a a)) : pt c d a :=
    dc.dc₂ (dcpt₃ h₁)
```

- dc_3^{pt}

```
theorem dc₃_pt {a b c d: Prop} (h₁ : pt c d a) : pt c d (dc b a a) :=
    dcpt₄ (dc.dc₃ h₁)
```

- dc_4^{pt}

```
theorem dc₄_pt {a b c d e f g : Prop} (h₁ : pt f g (dc d e (dc a b c))) :
    pt f g (dc e d (dc b a c)) :=
    have h₂ : dc (pt f g d) (pt f g e) (pt f g (dc a b c)), from dcpt₃ h₁,
    have h₃ : dc (pt f g d) (pt f g e) (dc (pt f g a) (pt f g b) (pt f g c)), from dcpt₅ h₂,
    have h₄ : dc (pt f g e) (pt f g d) (dc (pt f g b) (pt f g a) (pt f g c)), from dc.dc₄ h₃,
    have h₅ : dc (pt f g e) (pt f g d) (pt f g (dc b a c)), from dcpt₆ h₄,
    show pt f g (dc e d (dc b a c)), from dcpt₄ h₅
```

- dc_5^{pt}

```
theorem dc₅_pt {a b c d e f g : Prop} (h₁ : pt f g (dc d e (dc a b c))) :
    pt f g (dc e d (dc a c b)) :=
    have h₂ : dc (pt f g d) (pt f g e) (pt f g (dc a b c)), from dcpt₃ h₁,
    have h₃ : dc (pt f g d) (pt f g e) (dc (pt f g a) (pt f g b) (pt f g c)), from dcpt₅ h₂,
    have h₄ : dc (pt f g e) (pt f g d) (dc (pt f g a) (pt f g c) (pt f g b)), from dc.dc₅ h₃,
```

```
  have h₅ : dc (pt f g e) (pt f g d) (pt f g (dc a c b)), from dcpt₆ h₄,
  show pt f g (dc e d (dc a c b)), from dcpt₄ h₅
```

- $\text{dcpt}_5^{\text{dc}}$

```
theorem dcpt₅_dc {a b c d e f g h i : Prop} (h₁ : dc h i (dc f g (pt a b (dc c d e)))) :
  dc h i (dc f g (dc (pt a b c) (pt a b d) (pt a b e))) :=
  dc.dc₇' (dcpt₅ (dc.dc₆' h₁))
```

- $\text{dcpt}_6^{\text{dc}}$

```
theorem dcpt₆_dc {a b c d e f g h i : Prop}
  (h₁ : dc h i (dc f g (dc (pt a b c) (pt a b d) (pt a b e)))) :
  dc h i (dc f g (pt a b (dc c d e))) :=
  dc.dc₇' (dcpt₆ (dc.dc₆' h₁))
```

- dc_6^{pt}

```
theorem dc₆_pt {a b c d e f g h i : Prop}
  (h₁ : pt h i (dc f g (dc d e (dc a b c)))) : pt h i (dc f g (dc (dc d e a) (dc d e b) c)) :=
  let f' := pt h i f, g' := pt h i g, d' := pt h i d, e' := pt h i e in
    have h₂ : dc f' g' (pt h i (dc d e (dc a b c))),
      from dcpt₃ h₁,
    have h₃ : dc f' g' (dc d' e' (pt h i (dc a b c))),
      from dcpt₅ h₂,
    have h₄ : dc (dc f' g' d') (dc f' g' e') ((pt h i (dc a b c))),
      from dc.dc₆' h₃,
    have h₅ : dc (dc f' g' d') (dc f' g' e') (dc (pt h i a) (pt h i b) (pt h i c)),
      from dcpt₅ h₄,
    have h₆ : dc f' g' (dc d' e' (dc (pt h i a) (pt h i b) (pt h i c))),
      from dc.dc₇' h₅,
    have h₇ : dc f' g' (dc (dc d' e' (pt h i a)) (dc d' e' (pt h i b)) (pt h i c)),
      from dc.dc₆ h₆,
    have h₈ : dc g' f' (dc (dc d' e' (pt h i a)) (pt h i c) (dc d' e' (pt h i b))),
      from dc.dc₅ h₇,
    have h₉ : dc g' f' (dc (dc d' e' (pt h i a)) (pt h i c) (pt h i (dc d e b))),
      from dcpt₆_dc h₈,
    have h₁₀ : dc g' f' (dc (pt h i c) (pt h i (dc d e b)) (dc d' e' (pt h i a))),
      from dc.dc₅ (dc.dc₄ h₉),
    have h₁₁ : dc g' f' (dc (pt h i c) (pt h i (dc d e b)) (pt h i (dc d e a))),
      from dcpt₆_dc h₁₀,
```

```
    have h₁₂ : dc f' g' (dc (pt h i (dc d e a)) (pt h i (dc d e b)) (pt h i c)),
      from dc.dc₅ (dc.dc₄ (dc.dc₅ h₁₁)),
    have h₁₃ : dc f' g' (pt h i (dc (dc d e a) (dc d e b) c)),
      from dcpt₆ h₁₂,
    show pt h i (dc f g (dc (dc d e a) (dc d e b) c)),
      from dcpt₄ h₁₃
```

- dc_7^{pt}

```
theorem dc₇_pt {a b c d e f g h i: Prop}
  (h₁ : pt h i (dc f g (dc (dc d e a) (dc d e b) c))) : pt h i (dc f g (dc d e (dc a b c))) :=
  let f' := pt h i f, g' := pt h i g, d' := pt h i d, e' := pt h i e in
    have h₂ : dc f' g' (pt h i (dc (dc d e a) (dc d e b) c)),
      from dcpt₃ h₁,
    have h₃ : dc f' g' (dc (pt h i (dc d e a)) (pt h i (dc d e b)) (pt h i c)),
      from dcpt₅ h₂,
    have h₄ : dc g' f' (dc (pt h i (dc d e a)) (pt h i c) (pt h i (dc d e b))),
      from dc.dc₅ h₃,
    have h₅ : dc g' f' (dc (pt h i (dc d e a)) (pt h i c) (dc d' e' (pt h i b))),
      from dcpt₅_dc h₄,
    have h₆ : dc g' f' (dc (pt h i c) (dc d' e' (pt h i b)) (pt h i (dc d e a))),
      from dc.dc₅ (dc.dc₄ h₅),
    have h₇ : dc g' f' (dc (pt h i c) (dc d' e' (pt h i b)) (dc d' e' (pt h i a))),
      from dcpt₅_dc h₆,
    have h₈ : dc f' g' (dc (dc d' e' (pt h i a)) (dc d' e' (pt h i b)) (pt h i c)),
      from dc.dc₅ (dc.dc₄ (dc.dc₅ h₇)),
    have h₉ : dc f' g' (dc d' e' (dc (pt h i a) (pt h i b) (pt h i c))),
      from dc.dc₇ h₈,
    have h₁₀ : dc f' g' (dc d' e' (pt h i (dc a b c))),
      from dcpt₆_dc h₉,
    have h₁₁ : dc f' g' (pt h i (dc d e (dc a b c))),
      from dcpt₆ h₁₀,
    show pt h i (dc f g (dc d e (dc a b c))),
      from dcpt₄ h₁₁
```

- $dc_4'^{pt}$

```
theorem dc₄'_pt {a b c d e : Prop} (h₁ : pt d e (dc a b c)) : pt d e (dc b a c) :=
  have h₂ : pt d e (dc (dc b a c) (dc a b c) (dc a b c)), from dc₃_pt h₁,
  have h₃ : pt d e (dc (dc a b c) (dc b a c) (dc b a c)), from dc₄_pt h₂,
  show pt d e (dc b a c), from dc₂_pt h₃
```

- dc'^{pt}_5

```
theorem dc₅'_pt {a b c d e : Prop} (h₁ : pt d e (dc a b c)) : pt d e (dc a c b) :=
  have h₂ : pt d e (dc (dc a c b) (dc a b c) (dc a b c)), from dc₃_pt h₁,
  have h₃ : pt d e (dc (dc a b c) (dc a c b) (dc a c b)), from dc₅_pt h₂,
  show pt d e (dc a c b), from dc₂_pt h₃
```

- dc'^{pt}_6

```
theorem dc₆'_pt {a b c d e f g : Prop}
  (h₁ : pt f g (dc d e (dc a b c))) : pt f g (dc (dc d e a) (dc d e b) c) :=
  let h := dc d e (dc a b c), i := dc (dc d e a) (dc d e b) c in
    have h₂ : pt f g (dc i h h), from dc₃_pt h₁,
    have h₃ : pt f g (dc i h i), from dc₆_pt h₂,
    have h₄ : pt f g (dc h i i), from dc₄'_pt h₃,
    show pt f g i, from dc₂_pt h₄
```

- dc'^{pt}_7

```
theorem dc₇'_pt {a b c d e f g : Prop}
  (h₁ : pt f g (dc (dc d e a) (dc d e b) c)) : pt f g (dc d e (dc a b c)) :=
  let h := dc d e (dc a b c), i := dc (dc d e a) (dc d e b) c in
    have h₂ : pt f g (dc h i i), from dc₃_pt h₁,
    have h₃ : pt f g (dc h i h), from dc₇_pt h₂,
    have h₄ : pt f g (dc i h h), from dc₄'_pt h₃,
    show pt f g h, from dc₂_pt h₄
```

- $dcpt^{dc}_1$

```
theorem dcpt₁_dc {a b c d e f g : Prop} (h₁ : dc f g (dc a b (pt c d e))) :
  dc f g (pt (dc a b c) (dc a b d) (dc a b e)) :=
  let a' := dc f g a, b' := dc f g b, d' := dc a b d, e' := dc a b e in
    have h₁ : dc a' b' (pt c d e),
      from dc.dc₆' h₁,
    have h₂ : pt (dc a' b' c) (dc a' b' d) (dc a' b' e),
      from dcpt₁ h₁,
    have h₃ : pt (dc a' b' c) (dc a' b' d) (dc f g e'),
      from dc₇'_pt h₂,
    have h₄ : pt (dc a' b' c) (dc f g e') (dc a' b' d),
```

```
      from pt.pt₃ h₃,
    have h₅ : pt (dc a' b' c) (dc f g e') (dc f g d'),
      from dc₇'_pt h₄,
    have h₆ : pt (dc f g e') (dc f g d') (dc a' b' c),
      from pt.pt₃ (pt.pt₂ h₅),
    have h₇ : pt (dc f g e') (dc f g d') (dc f g (dc a b c)),
      from dc₇'_pt h₆,
    have h₈ : pt (dc f g (dc a b c)) (dc f g d') (dc f g e'),
      from pt.pt₂ (pt.pt₃ (pt.pt₂ h₇)),
    show dc f g (pt (dc a b c) d' e'),
      from dcpt₂ h₈
```

- $\mathsf{dcpt}_2^{\mathsf{dc}}$

```
theorem dcpt₂_dc {a b c d e f g : Prop} (h₁ : dc f g (pt (dc a b c) (dc a b d) (dc a b e))) :
  dc f g (dc a b (pt c d e)) :=
  let a' := dc f g a, b' := dc f g b, d' := dc a b d, e' := dc a b e in
    have h₂ : pt (dc f g (dc a b c)) (dc f g d') (dc f g e'),
      from dcpt₁ h₁,
    have h₃ : pt (dc f g (dc a b c)) (dc f g d') (dc a' b' e),
      from dc₆'_pt h₂,
    have h₄ : pt (dc f g (dc a b c)) (dc a' b' e) (dc f g d'),
      from pt.pt₃ h₃,
    have h₅ : pt (dc f g (dc a b c)) (dc a' b' e) (dc a' b' d),
      from dc₆'_pt h₄,
    have h₆ : pt (dc a' b' e) (dc a' b' d) (dc f g (dc a b c)),
      from pt.pt₃ (pt.pt₂ h₅),
    have h₇ : pt (dc a' b' e) (dc a' b' d) (dc a' b' c),
      from dc₆'_pt h₆,
    have h₈ : pt (dc a' b' c) (dc a' b' d) (dc a' b' e),
      from pt.pt₂ (pt.pt₃ (pt.pt₂ h₇)),
    have h₉ : dc a' b' (pt c d e),
      from dcpt₂ h₈,
    show dc f g (dc a b (pt c d e)),
      from dc.dc₇' h₉
```

- $\mathsf{dcpt}_3^{\mathsf{dc}}$

```
theorem dcpt₃_dc {a b c d e f g : Prop} (h₁ : dc f g (pt a b (dc c d e))) :
  dc f g (dc (pt a b c) (pt a b d) (pt a b e)) := dcpt₅ h₁
```

- $\mathsf{dcpt}_4^{\mathsf{dc}}$

```
theorem dcpt4_dc {a b c d e f g : Prop} (h1 : dc f g (dc (pt a b c) (pt a b d) (pt a b e))) :
  dc f g (pt a b (dc c d e)) := dcpt6 h1
```

- $\mathsf{dcpt}_7^{\mathsf{dc}}$

```
theorem dcpt7_dc {a b c d e f g h i : Prop} (h1 : dc h i (pt f g (dc a b (pt c d e)))) :
  dc h i (pt f g (pt (dc a b c) (dc a b d) (dc a b e))) :=
  let f' := dc h i f, g' := dc h i g, a' := dc h i a, b' := dc h i b in
    have h2 : pt f' g' (dc h i (dc a b (pt c d e))),
      from dcpt1 h1,
    have h3 : pt f' g' (dc a' b' (pt c d e)),
      from dc6'_pt h2,
    have h4 : pt f' g' (pt (dc a' b' c) (dc a' b' d) (dc a' b' e)),
      from dcpt7 h3,
    have h5 : pt (pt f' g' (dc a' b' c)) (dc a' b' d) (dc a' b' e),
      from pt.pt6 h4,
    have h6 : pt (pt f' g' (dc a' b' c)) (dc a' b' d) (dc h i (dc a b e)),
      from dc7'_pt h5,
    have h7 : pt (pt f' g' (dc a' b' c)) (dc h i (dc a b e)) (dc a' b' d),
      from pt.pt3 h6,
    have h8 : pt (pt f' g' (dc a' b' c)) (dc h i (dc a b e)) (dc h i (dc a b d)),
      from dc7'_pt h7,
    have h9 : pt f' g' (pt (dc a' b' c) (dc h i (dc a b e)) (dc h i (dc a b d))),
      from pt.pt7 h8,
    have h10 : pt f' g' (pt (dc h i (dc a b e)) (dc h i (dc a b d)) (dc a' b' c)),
      from pt.pt3_ast (pt.pt2_ast h9),
    have h11 : pt (pt f' g' (dc h i (dc a b e))) (dc h i (dc a b d)) (dc a' b' c),
      from pt.pt6 h10,
    have h12 : pt (pt f' g' (dc h i (dc a b e))) (dc h i (dc a b d)) (dc h i (dc a b c)),
      from dc7'_pt h11,
    have h13 : pt f' g' (pt (dc h i (dc a b e)) (dc h i (dc a b d)) (dc h i (dc a b c))),
      from pt.pt7 h12,
    have h14 : pt f' g' (pt (dc h i (dc a b c)) (dc h i (dc a b d)) (dc h i (dc a b e))),
      from pt.pt2_ast (pt.pt3_ast (pt.pt2_ast h13)),
    have h15 : pt f' g' (dc h i (pt (dc a b c) (dc a b d) (dc a b e))),
      from dcpt8 h14,
    show dc h i (pt f g (pt (dc a b c) (dc a b d) (dc a b e))),
      from dcpt2 h15
```

- $\mathsf{dcpt}_8^{\mathsf{dc}}$

```
theorem dcpt₈_dc {a b c d e f g h i : Prop}
  (h₁ : dc h i (pt f g (pt (dc a b c) (dc a b d) (dc a b e)))) :
  dc h i (pt f g (dc a b (pt c d e))) :=
  let f' := dc h i f, g' := dc h i g, a' := dc h i a, b' := dc h i b in
    have h₂ : pt f' g' (dc h i (pt (dc a b c) (dc a b d) (dc a b e))),
      from dcpt₁ h₁,
    have h₃ : pt f' g' (pt (dc h i (dc a b c)) (dc h i (dc a b d)) (dc h i (dc a b e))),
      from dcpt₇ h₂,
    have h₄ : pt (pt f' g' (dc h i (dc a b c))) (dc h i (dc a b d)) (dc h i (dc a b e)),
      from pt.pt₆ h₃,
    have h₅ : pt (pt f' g' (dc h i (dc a b c))) (dc h i (dc a b d)) (dc a' b' e),
      from dc₆'_pt h₄,
    have h₆ : pt (pt f' g' (dc h i (dc a b c))) (dc a' b' e) (dc h i (dc a b d)),
      from pt.pt₃ h₅,
    have h₇ : pt (pt f' g' (dc h i (dc a b c))) (dc a' b' e) (dc a' b' d),
      from dc₆'_pt h₆,
    have h₈ : pt f' g' (pt (dc h i (dc a b c)) (dc a' b' e) (dc a' b' d)),
      from pt.pt₇ h₇,
    have h₉ : pt f' g' (pt (dc a' b' e) (dc a' b' d) (dc h i (dc a b c))),
      from pt.pt₃_pt (pt.pt₂_pt h₈),
    have h₁₀ : pt (pt f' g' (dc a' b' e)) (dc a' b' d) (dc h i (dc a b c)),
      from pt.pt₆ h₉,
    have h₁₁ : pt (pt f' g' (dc a' b' e)) (dc a' b' d) (dc a' b' c),
      from dc₆'_pt h₁₀,
    have h₁₂ : pt f' g' (pt (dc a' b' e) (dc a' b' d) (dc a' b' c)),
      from pt.pt₇ h₁₁,
    have h₁₃ : pt f' g' (pt (dc a' b' c) (dc a' b' d) (dc a' b' e)),
      from pt.pt₂_pt (pt.pt₃_pt (pt.pt₂_pt h₁₂)),
    have h₁₄ : pt f' g' (dc a' b' (pt c d e)),
      from dcpt₈ h₁₃,
    have h₁₅ : pt f' g' (dc h i (dc a b (pt c d e))),
      from dc₇'_pt h₁₄,
    show dc h i (pt f g (dc a b (pt c d e))),
      from dcpt₂ h₁₅
```

- $\mathsf{dcpt}_1^{\mathsf{pt}}$

```
theorem dcpt₁_pt {a b c d e f g : Prop} (h₁ : pt f g (dc a b (pt c d e))) :
  pt f g (pt (dc a b c) (dc a b d) (dc a b e)) := dcpt₇ h₁
```

- $\mathsf{dcpt}_2^{\mathsf{pt}}$

```
theorem dcpt₂_pt {a b c d e f g : Prop} (h₁ : pt f g (pt (dc a b c) (dc a b d) (dc a b e))) :
  pt f g (dc a b (pt c d e)) := dcpt₈ h₁
```

- $\mathsf{dcpt}_3^{\mathsf{pt}}$

```
theorem dcpt₃_pt {a b c d e f g : Prop}
  (h₁ : pt f g (pt a b (dc c d e))) :
  pt f g (dc (pt a b c) (pt a b d) (pt a b e)) :=
  let a' := pt f g a, c' := pt a b c, d' := pt a b d, e' := pt a b e in
    have h₂ : pt a' b (dc c d e), from pt.pt₆ h₁,
    have h₃ : dc (pt a' b c) (pt a' b d) (pt a' b e), from dcpt₃ h₂,
    have h₄ : dc (pt a' b c) (pt a' b d) (pt f g e'), from pt₇_dc h₃,
    have h₅ : dc (pt a' b c) (pt f g e') (pt a' b d), from dc.dc₅' h₄,
    have h₆ : dc (pt a' b c) (pt f g e') (pt f g d'), from pt₇_dc h₅,
    have h₇ : dc (pt f g e') (pt f g d') (pt a' b c), from dc.dc₅' (dc.dc₄' h₆),
    have h₈ : dc (pt f g e') (pt f g d') (pt f g c'), from pt₇_dc h₇,
    have h₉ : dc (pt f g c') (pt f g d') (pt f g e'), from dc.dc₄' (dc.dc₅' (dc.dc₄' h₈)),
    show pt f g (dc c' d' e'), from dcpt₄ h₉
```

- $\mathsf{dcpt}_4^{\mathsf{pt}}$

```
theorem dcpt₄_pt {a b c d e f g : Prop} (h₁ : pt f g (dc (pt a b c) (pt a b d) (pt a b e))) :
  pt f g (pt a b (dc c d e)) :=
  let a' := pt f g a, c' := pt a b c, d' := pt a b d, e' := pt a b e in
    have h₂ : dc (pt f g c') (pt f g (pt a b d)) (pt f g e'), from dcpt₃ h₁,
    have h₃ : dc (pt f g c') (pt f g (pt a b d)) (pt a' b e), from pt₆_dc h₂,
    have h₄ : dc (pt f g c') (pt a' b e) (pt f g (pt a b d)), from dc.dc₅' h₃,
    have h₅ : dc (pt f g c') (pt a' b e) (pt a' b d), from pt₆_dc h₄,
    have h₆ : dc (pt a' b e) (pt a' b d) (pt f g c'), from dc.dc₅' (dc.dc₄' h₅),
    have h₇ : dc (pt a' b e) (pt a' b d) (pt a' b c), from pt₆_dc h₆,
    have h₈ : dc (pt a' b c) (pt a' b d) (pt a' b e), from dc.dc₄' (dc.dc₅' (dc.dc₄' h₇)),
    have h₉ : pt a' b (dc c d e), from dcpt₄ h₈,
    show pt f g (pt a b (dc c d e)), from pt.pt₇ h₉
```

- $\mathsf{dcpt}_5^{\mathsf{pt}}$

```
theorem dcpt₅_pt {a b c d e f g h i : Prop} (h₁ : pt h i (dc f g (pt a b (dc c d e)))) :
  pt h i (dc f g (dc (pt a b c) (pt a b d) (pt a b e))) :=
  let f' := pt h i f, g' := pt h i g, a' := pt h i a, d' := pt a b d, e' := pt a b e in
```

```
    have h₂ : dc f' g' (pt h i (pt a b (dc c d e))),
      from dcpt₃ h₁,
    have h₃ : dc f' g' (pt a' b (dc c d e)),
      from pt₆_dc h₂,
    have h₄ : dc f' g' (dc (pt a' b c) (pt a' b d) (pt a' b e)),
      from dcpt₅ h₃,
    have h₅ : dc (dc f' g' (pt a' b c)) (dc f' g' (pt a' b d)) (pt a' b e),
      from dc.dc₆' h₄,
    have h₆ : dc (dc f' g' (pt a' b c)) (dc f' g' (pt a' b d)) (pt h i e'),
      from pt₇_dc h₅,
    have h₇ : dc f' g' (dc (pt a' b c) (pt a' b d) (pt h i e')),
      from dc.dc₇' h₆,
    have h₈ : dc g' f' (dc (pt a' b c) (pt h i e') (pt a' b d)),
      from dc.dc₅ h₇,
    have h₉ : dc (dc g' f' (pt a' b c)) (dc g' f' (pt h i e')) (pt a' b d),
      from dc.dc₆' h₈,
    have h₁₀ : dc (dc g' f' (pt a' b c)) (dc g' f' (pt h i e')) (pt h i d'),
      from pt₇_dc h₉,
    have h₁₁ : dc g' f' (dc (pt a' b c) (pt h i e') (pt h i d')),
      from dc.dc₇' h₁₀,
    have h₁₂ : dc g' f' (dc (pt h i e') (pt h i d') (pt a' b c)),
      from dc.dc₅ (dc.dc₄ h₁₁),
    have h₁₃ : dc (dc g' f' (pt h i e')) (dc g' f' (pt h i d')) (pt a' b c),
      from dc.dc₆' h₁₂,
    have h₁₄ : dc (dc g' f' (pt h i e')) (dc g' f' (pt h i d')) (pt h i (pt a b c)),
      from pt₇_dc h₁₃,
    have h₁₅ : dc g' f' (dc (pt h i e') (pt h i d') (pt h i (pt a b c))),
      from dc.dc₇' h₁₄,
    have h₁₆ : dc f' g' (dc (pt h i (pt a b c)) (pt h i d') (pt h i e')),
      from dc.dc₄ (dc.dc₅ (dc.dc₄ h₁₅)),
    have h₁₇ : dc f' g' (pt h i (dc (pt a b c) d' e')),
      from dcpt₆ h₁₆,
    show pt h i (dc f g (dc (pt a b c) d' e')),
      from dcpt₄ h₁₇
```

- $\mathsf{dcpt}_6^{\mathsf{pt}}$

```
theorem dcpt₆_pt {a b c d e f g h i : Prop}
  (h₁ : pt h i (dc f g (dc (pt a b c) (pt a b d) (pt a b e)))) :
  pt h i (dc f g (pt a b (dc c d e))) :=
  let f' := pt h i f, g' := pt h i g, a' := pt h i a, d' := pt a b d, e' := pt a b e in
    have h₂ : dc f' g' (pt h i (dc (pt a b c) d' e')),
```

```
    from dcpt₃ h₁,
  have h₃ : dc f' g' (dc (pt h i (pt a b c)) (pt h i d') (pt h i e')),
    from dcpt₅ h₂,
  have h₄ : dc g' f' (dc (pt h i e') (pt h i d') (pt h i (pt a b c))),
    from dc.dc₄ (dc.dc₅ (dc.dc₄ h₃)),
  have h₅ : dc (dc g' f' (pt h i e')) (dc g' f' (pt h i d')) (pt h i (pt a b c)),
    from dc.dc₆' h₄,
  have h₆ : dc (dc g' f' (pt h i e')) (dc g' f' (pt h i d')) (pt a' b c),
    from pt₆_dc h₅,
  have h₇ : dc g' f' (dc (pt h i e') (pt h i d') (pt a' b c)),
    from dc.dc₇' h₆,
  have h₈ : dc g' f' (dc (pt a' b c) (pt h i e') (pt h i d')),
    from dc.dc₄ (dc.dc₅ h₇),
  have h₉ : dc (dc g' f' (pt a' b c)) (dc g' f' (pt h i e')) (pt h i d'),
    from dc.dc₆' h₈,
  have h₁₀ : dc (dc g' f' (pt a' b c)) (dc g' f' (pt h i e')) (pt a' b d),
    from pt₆_dc h₉,
  have h₁₁ : dc g' f' (dc (pt a' b c) (pt h i e') (pt a' b d)),
    from dc.dc₇' h₁₀,
  have h₁₂ : dc f' g' (dc (pt a' b c) (pt a' b d) (pt h i e')),
    from dc.dc₅ h₁₁,
  have h₁₃ : dc (dc f' g' (pt a' b c)) (dc f' g' (pt a' b d)) (pt h i e'),
    from dc.dc₆' h₁₂,
  have h₁₄ : dc (dc f' g' (pt a' b c)) (dc f' g' (pt a' b d)) (pt a' b e),
    from pt₆_dc h₁₃,
  have h₁₅ : dc f' g' (dc (pt a' b c) (pt a' b d) (pt a' b e)),
    from dc.dc₇' h₁₄,
  have h₁₆ : dc f' g' (pt a' b (dc c d e)),
    from dcpt₆ h₁₅,
  have h₁₇ : dc f' g' (pt h i (pt a b (dc c d e))),
    from pt₇_dc h₁₆,
  show pt h i (dc f g (pt a b (dc c d e))),
    from dcpt₄ h₁₇
```

- $\mathsf{dcpt}_7^{\mathsf{pt}}$

```
theorem dcpt₇_pt {a b c d e f g h i : Prop} (h₁ : pt h i (pt f g (dc a b (pt c d e)))) :
  pt h i (pt f g (pt (dc a b c) (dc a b d) (dc a b e))) :=
  have h₂ : pt (pt h i f) g (dc a b (pt c d e)), from pt.pt₆ h₁,
  have h₃ : pt (pt h i f) g (pt (dc a b c) (dc a b d) (dc a b e)), from dcpt₇ h₂,
  show pt h i (pt f g (pt (dc a b c) (dc a b d) (dc a b e))), from pt.pt₇ h₃
```

- $\mathsf{dcpt}_8^{\mathsf{pt}}$

```
theorem dcpt₈_pt {a b c d e f g h i : Prop}
  (h₁ : pt h i (pt f g (pt (dc a b c) (dc a b d) (dc a b e)))) :
  pt h i (pt f g (dc a b (pt c d e))) :=
  have h₂ : pt (pt h i f) g (pt (dc a b c) (dc a b d) (dc a b e)), from pt.pt₆ h₁,
  have h₃ : pt (pt h i f) g (dc a b (pt c d e)), from dcpt₈ h₂,
  show pt h i (pt f g (dc a b (pt c d e))), from pt.pt₇ h₃
```

□

Theorem 4.12.3. *The calculus $\mathscr{B}_{\mathsf{pt,dc}}$ is complete with respect to the matrix $\mathbb{2}_{\mathsf{pt,dc}}$.*

Proof. Notice that rules $\mathsf{dc}_j^{\mathsf{pt}}$, $\mathsf{pt}_i^{\mathsf{pt}}$, and $\mathsf{dcpt}_k^{\mathsf{pt}}$, where $1 \leq \mathsf{i} \leq 6$, $1 \leq \mathsf{j} \leq 7$ and $1 \leq \mathsf{k} \leq 8$, are all provable in $\mathscr{B}_{\mathsf{dc,pt}}$, by Lemma 4.9.2 and Lemma 4.12.2. This fact implies that the property m_{pt}, and thus the completeness property (pt), hold in $\mathscr{B}_{\mathsf{pt,dc}}$, by Remark 4.9.1. A similar argument justifies the preservation of the completeness property (dc), in view of Remark 4.11.1: use the fact that rules $\mathsf{pt}_i^{\mathsf{dc}}$, $\mathsf{dc}_j^{\mathsf{dc}}$, and $\mathsf{dcpt}_k^{\mathsf{dc}}$, where $1 \leq \mathsf{i} \leq 6$, $1 \leq \mathsf{j} \leq 7$ and $1 \leq \mathsf{k} \leq 8$, are all provable in $\mathscr{B}_{\mathsf{pt,dc}}$, by Lemma 4.11.2 and Lemma 4.12.2. □

4.13 $\mathcal{B}_{\mathsf{dc},\neg}$

We present now a calculus for the fragment containing in its signature only dc and $\neg$. The purpose is to extend $\mathscr{B}_{\mathsf{dc}}$ with only two interaction rules, which are proved sound with respect to $\mathbb{2}_{\mathsf{dc},\neg}$ right after the presentation of $\mathscr{B}_{\mathsf{dc},\neg}$ below.

Hilbert Calculus 31. $\mathscr{B}_{\mathsf{dc},\neg}$

$$\mathscr{B}_{\mathsf{dc}} \qquad \frac{\mathsf{dc}(C, D, \mathsf{dc}(B, A, \neg A))}{\mathsf{dc}(C, D, B)}\ \mathsf{dcn}_1 \qquad \frac{\mathsf{dc}(C, D, B)}{\mathsf{dc}(C, D, \mathsf{dc}(B, A, \neg A))}\ \mathsf{dcn}_2$$

Theorem 4.13.1. *The calculus $\mathscr{B}_{\mathsf{dc},\neg}$ is sound with respect to the matrix $\mathbb{2}_{\mathsf{dc},\neg}$.*

Proof. Let v be an arbitrary $\mathbb{2}_{\mathsf{dc},\neg}$-valuation. The soundness result for the rules of $\mathscr{B}_{\mathsf{dc}}$ was already proved in Theorem 4.11.1. For dcn_1, suppose that v assigns 0 to its conclusion. Then we have the following possibilities:

- $v(\mathrm{C}) = 1$, $v(\mathrm{D}) = 0$ and $v(\mathrm{B}) = 0$: since $v(\mathrm{B}) = 0$, $v(\mathsf{dc}(\mathrm{B}, \mathrm{A}, \neg\mathrm{A})) = 0$, no matter the value of $v(\mathrm{A})$, then $v(\mathsf{dc}(\mathrm{C}, \mathrm{D}, \mathsf{dc}(\mathrm{B}, \mathrm{A}, \neg\mathrm{A}))) = 0$.

- $v(\mathrm{C}) = 0$, $v(\mathrm{D}) = 1$ and $v(\mathrm{B}) = 0$: analogous to the previous case.

- $v(\mathrm{C}) = 0$, $v(\mathrm{D}) = 0$ and $v(\mathrm{B}) = 1$: since $v(\mathrm{B}) = 1$, $v(\mathsf{dc}(\mathrm{B}, \mathrm{A}, \neg\mathrm{A})) = 1$, no matter the value of $v(\mathrm{A})$, then $v(\mathsf{dc}(\mathrm{C}, \mathrm{D}, \mathsf{dc}(\mathrm{B}, \mathrm{A}, \neg\mathrm{A}))) = 0$, because $v(\mathrm{C}) = v(\mathrm{D}) = 0$.

- $v(\mathrm{C}) = 0$, $v(\mathrm{D}) = 0$ and $v(\mathrm{B}) = 0$: analogous to the previous case.

The proof for dcn_2 is similar by considering the cases in which $\mathsf{dc}(\mathrm{C}, \mathrm{D}, \mathrm{B})$ is evaluated to 1. □

We proceed now to derive some rules in $\mathscr{B}_{\mathsf{dc},\neg}$ that will be used in the completeness proof of this calculus with respect to $\mathfrak{Q}_{\mathsf{dc},\neg}$.

Lemma 4.13.2. *The following rules are derivable in* $\mathscr{B}_{\mathsf{dc},\neg}$*:*

$$\frac{\mathsf{dc}(\mathrm{E}, \mathrm{F}, \mathsf{dc}(\mathrm{C}, \mathrm{D}, \mathsf{dc}(\mathrm{B}, \mathrm{A}, \neg\mathrm{A})))}{\mathsf{dc}(\mathrm{E}, \mathrm{F}, \mathsf{dc}(\mathrm{C}, \mathrm{D}, \mathrm{B}))}\ \mathsf{dcn}_1^{\mathsf{dc}}$$

$$\frac{\mathsf{dc}(\mathrm{E}, \mathrm{F}, \mathsf{dc}(\mathrm{C}, \mathrm{D}, \mathrm{B}))}{\mathsf{dc}(\mathrm{E}, \mathrm{F}, \mathsf{dc}(\mathrm{C}, \mathrm{D}, \mathsf{dc}(\mathrm{B}, \mathrm{A}, \neg\mathrm{A})))}\ \mathsf{dcn}_2^{\mathsf{dc}}$$

$$\frac{\mathrm{A} \quad \neg\mathrm{A}}{\mathrm{B}}\ \mathsf{n}_1$$

$$\frac{\mathrm{B}}{\mathsf{dc}(\mathrm{B}, \mathrm{A}, \neg\mathrm{A})}\ \mathsf{dcn}_3$$

Proof. The formally verified derivation of each rule is presented below, following what was explained in Chapter 3.

- $\mathsf{dcn}_1^{\mathsf{dc}}$

```
theorem dcn₁_dc {a b c d e f : Prop} (h₁ : dc e f (dc c d (dc b a (neg a))))
  : dc e f (dc c d b) :=
  have h₂ : dc (dc e f c) (dc e f d) (dc b a (neg a)), from dc.dc₆' h₁,
  have h₃ : dc (dc e f c) (dc e f d) b, from dcn₁ h₂,
  show dc e f (dc c d b), from dc.dc₇' h₃
```

- $\mathsf{dcn}_2^{\mathsf{dc}}$

```
theorem dcn₂_dc {a b c d e f : Prop} (h₁ : dc e f (dc c d b))
  : dc e f (dc c d (dc b a (neg a))) :=
  have h₂ : dc (dc e f c) (dc e f d) b, from dc.dc₆' h₁,
  have h₃ : dc (dc e f c) (dc e f d) (dc b a (neg a)), from dcn₂ h₂,
  show dc e f (dc c d (dc b a (neg a))), from dc.dc₇' h₃
```

- n_1

```
theorem n₁ {a b : Prop} (h₁ : a) (h₂ : neg a) : b :=
  have h₂ : dc a (neg a) b, from dc.dc₁ h₁ h₂,
  have h₃ : dc (dc a (neg a) b) a b, from dc.dc₁ h₂ h₁,
  have h₄ : dc a b (dc a (neg a) b), from dc.dc₅' (dc.dc₄' h₃),
  have h₅ : dc a b (dc b a (neg a)), from dc.dc₄ (dc.dc₅ h₄),
  have h₆ : dc a b b, from dcn₁ h₅,
  show b, from dc.dc₂ h₆
```

- dcn_3

```
theorem dcn₃ {a b : Prop} (h₁ : b) : dc b a (neg a) :=
  have h₂ : dc a b b, from dc.dc₃ h₁,
  have h₃ : dc a b (dc b a (neg a)), from dcn₂ h₂,
  have h₄ : dc (dc a b b) (dc a b a) (neg a), from dc.dc₆' h₃,
  have h₅ : dc (neg a) (dc a b b) (dc a b a), from dc.dc₄' (dc.dc₅' h₄),
  have h₆ : dc (neg a) (dc a b b) (dc b a a), from dc.dc₅ (dc.dc₄ h₅),
  have h₇ : dc (neg a) (dc a b b) a, from dc.dc₂_dc h₆,
  have h₈ : dc (neg a) a (dc a b b), from dc.dc₅' h₇,
  have h₉ : dc (neg a) a b, from dc.dc₂_dc h₈,
  show dc b a (neg a), from dc.dc₅' (dc.dc₄' (dc.dc₅' h₉))
```

□

Theorem 4.13.3. *The calculus $\mathscr{B}_{\mathsf{dc},\neg}$ is complete with respect to the matrix $\mathfrak{Q}_{\mathsf{dc},\neg}$.*

Proof. According to the procedure given in Section 2.7, we need to prove the completeness properties $(\neg)$ and (dc). Notice that the properties m_{dc} and δ_{dc} hold in this calculus, because the dc-lifted versions of the rules dcn_1 and dcn_2 are derivable $\mathscr{B}_{\mathsf{dc},\neg}$ (see Remark 4.11.1); therefore, (dc) also holds in $\mathscr{B}_{\mathsf{dc},\neg}$ (check the proof of Theorem 4.11.5). In order to finish this proof, the completeness property for $\neg$, namely $(\neg)$ $\neg A \in \Gamma^+$ iff $A \notin \Gamma^+$, must be proved. The left-to-right direction is proved in the same way as in the proof of completeness of the calculus $\mathscr{B}_{\neg}$, since n_1 is derivable in $\mathscr{B}_{\mathsf{dc},\neg}$. From the right to the left, the proof goes by contradiction: suppose that $A, \neg A \notin \Gamma^+$, so (a): $\Gamma^+, A \vdash_{\mathscr{B}_{\mathsf{dc},\neg}} Z$ and (b): $\Gamma^+, \neg A \vdash_{\mathscr{B}_{\mathsf{dc},\neg}} Z$. Because $\mathbb{2}_{\mathsf{dc},\neg}$ has no tautologies, Lemma 2.6.2 guarantees that Γ^+ is nonempty, so we can take some $B \in \Gamma^+$. Then, by δ_{dc}, from (a) and (b), we have (c): $\Gamma^+, \mathsf{dc}(B, A, \neg A) \vdash_{\mathscr{B}_{\mathsf{dc},\neg}} Z$. We also have, by rule dcn_3, (d): $B \vdash_{\mathscr{B}_{\mathsf{dc},\neg}} \mathsf{dc}(B, A, \neg A)$. So, by (T), from (c) and (d), we get $\Gamma^+, B \vdash_{\mathscr{B}_{\mathsf{dc},\neg}} Z$, but $B \in \Gamma^+$, thus $\Gamma^+ \vdash_{\mathscr{B}_{\mathsf{dc},\neg}} Z$, a contradiction. □

4.14 $\mathcal{B}_{\wedge,\vee}$, $\mathcal{B}_{\wedge,\vee,\top}$, $\mathcal{B}_{\wedge,\vee,\bot}$, $\mathcal{B}_{\wedge,\vee,\bot,\top}$

The calculus $\mathscr{B}_{\wedge,\vee}$ for the fragment $\mathcal{B}_{\wedge,\vee}$, presented below, is produced by adding to the calculus $\mathscr{B}_{\vee}$ some rules of interaction that will guarantee the preservation of the properties ($\wedge$) and ($\vee$), necessary for completeness.

Hilbert Calculus 32. $\mathscr{B}_{\wedge,\vee}$

$$\mathscr{B}_{\vee} \quad \frac{C \vee A \quad C \vee B}{C \vee (A \wedge B)}\,\mathsf{cd}_1 \qquad \frac{C \vee (A \wedge B)}{C \vee A}\,\mathsf{cd}_2 \qquad \frac{C \vee (A \wedge B)}{C \vee B}\,\mathsf{cd}_3$$

Theorem 4.14.1. *The calculus $\mathscr{B}_{\wedge,\vee}$ is sound with respect to the matrix $\mathbb{2}_{\wedge,\vee}$.*

Proof. Only soundness of cd_i, where $1 \leq i \leq 3$, remains to be proved. Let v be a $\mathbb{2}_{\wedge,\vee}$-evaluation. Notice that, if $v(C) = 1$, premisses and conclusions of these rules will be necessarily evaluated to 1. In case $v(C) = 0$, the argument is analogous to the one used in the proof of soundness for $\mathscr{B}_{\wedge}$ (see Theorem 4.2.1). □

Lemma 4.14.2. *The rules of $\mathscr{B}_{\wedge}$ are derivable in $\mathscr{B}_{\wedge,\vee}$.*

Proof. The formally verified derivation of each rule is presented below, following what was explained in Chapter 3.

- c_1

```
theorem c₁ {a b : Prop} (h₁ : a) (h₂ : b) : and a b :=
  have h₃ : or (and a b) a, from or.d₁' h₁,
  have h₄ : or (and a b) b, from or.d₁' h₂,
  have h₅ : or (and a b) (and a b), from cd₁ h₃ h₄,
```

```
    show and a b, from or.d₂ h₅
```

- c_2

```
theorem c₂ {a b : Prop} (h₁ : and a b) : a :=
  have h₂ : or a (and a b), from or.d₁' h₁,
  have h₃ : or a a, from cd₂ h₂,
  show a, from or.d₂ h₃
```

- c_3

```
theorem c₃ {a b : Prop} (h₁ : and a b) : b :=
  have h₂ : or b (and a b), from or.d₁' h₁,
  have h₃ : or b b, from cd₃ h₂,
  show b, from or.d₂ h₃
```

□

Theorem 4.14.3. *The calculus $\mathscr{B}_{\wedge,\vee}$ is complete with respect to the matrix $\mathbb{2}_{\wedge,\vee}$.*

Proof. Since the rules of $\mathscr{B}_{\wedge}$ are derivable in this calculus, as presented in Lemma 4.14.2, the completeness property $(\wedge)$ holds in $\mathscr{B}_{\wedge,\vee}$. In addition, because $\mathsf{cd}_i = \mathsf{c}_i^{\vee}$, for all $1 \leq i \leq 3$, $\mathsf{cd}_i^{\vee,2}$ is derivable in the proposed calculus by Lemma 4.5.7, so the completeness property $(\vee)$ also follows (check the proof of Theorem 4.5.6 for more details). □

Remark 4.14.1. The calculus $\mathscr{B}_{\wedge,\vee}$ would have been produced by the procedure implicit in the proof of Theorem 4.5.12 regarding the axiomatizability of monotonic expansions of $\mathcal{B}_{\vee}$.

The expansion $\mathcal{B}_{\wedge,\vee,\top}$ is directly axiomatized by the calculus below, in view of Corollary 2.8.4.1:

Hilbert Calculus 33. $\mathscr{B}_{\wedge,\vee,\top}$

$$\mathscr{B}_{\wedge,\vee} \quad \mathscr{B}_{\top}$$

The fragment $\mathcal{B}_{\wedge,\vee,\bot}$ is axiomatized by adding to the rules of $\mathcal{B}_{\wedge,\vee}$ the rule db_1 (as in $\mathscr{B}_{\vee,\bot}$), because the completeness property $(\wedge)$ is not affected by such modification (see Remark 4.2.1) and $\mathscr{B}_{\vee,\bot}$ is complete with respect to $\mathcal{B}_{\vee,\bot}$, as proved in Theorem 4.5.10.

Hilbert Calculus 34. $\mathscr{B}_{\wedge,\vee,\bot}$

$$\mathscr{B}_{\wedge,\vee} \qquad \frac{A \vee \bot}{A}\ \mathsf{db}_1$$

Finally, the axiomatization of $\mathcal{B}_{\wedge,\vee,\bot,\top}$ is another application of Corollary 2.8.4.1:

Hilbert Calculus 35. $\mathscr{B}_{\wedge,\vee,\bot,\top}$

$$\mathscr{B}_{\wedge,\vee,\bot} \qquad \mathscr{B}_{\top}$$

4.15 $\mathcal{B}_{\mathsf{ki},\vee}$, $\mathcal{B}_{\mathsf{ki},\vee,\bot}$, $\mathcal{B}_{\mathsf{ki},\vee,\top}$

The present fragments are at the top of Post's lattice and are expansions of $\mathcal{B}_{\mathsf{ki}}$. We will produce first an axiomatization for $\mathcal{B}_{\mathsf{ki},\vee}$ as a direct application of Theorem 4.4.9, which provides a procedure to axiomatize any expansion of $\mathcal{B}_{\mathsf{ki}}$ by adding some at most unary rules to $\mathscr{B}_{\mathsf{ki}}$.

Hilbert Calculus 36. $\mathscr{B}_{\mathsf{ki},\vee}$

$$\mathscr{B}_{\mathsf{ki}}$$

$$\frac{\mathsf{ki}(D, E, A \vee B)}{\mathsf{ki}(D, E, \mathsf{ki}(A \vee B, \mathsf{ki}(A \vee B, A, C), \mathsf{ki}(A \vee B, \mathsf{ki}(A \vee B, B, C), C)))}\ \mathsf{kid}_1$$

$$\frac{\mathsf{ki}(D, E, A)}{\mathsf{ki}(D, E, A \vee B)}\ \mathsf{kid}_2$$

$$\frac{\mathsf{ki}(D, E, B)}{\mathsf{ki}(D, E, A \vee B)}\ \mathsf{kid}_3$$

Next, in order to axiomatize $\mathcal{B}_{\mathsf{ki},\vee,\bot}$, we use again Theorem 4.4.9. The procedure now gives the calculus $\mathscr{B}_{\mathsf{ki},\vee}$ plus a rule of interaction to accomodate $\bot$.

Hilbert Calculus 37. $\mathscr{B}_{\mathsf{ki},\vee,\bot}$

$$\mathscr{B}_{\mathsf{ki},\vee} \quad \frac{\mathsf{ki}(\mathrm{A},\mathrm{B},\bot)}{\mathsf{ki}(\mathrm{A},\mathrm{B},\mathrm{C})}\ \mathsf{kidb}_1$$

Finally, we use Corollary 2.8.4.1 to axiomatize $\mathcal{B}_{\mathsf{ki},\vee,\top}$.

Hilbert Calculus 38. $\mathscr{B}_{\mathsf{ki},\vee,\top}$

$$\mathscr{B}_{\mathsf{ki},\vee} \quad \mathscr{B}_{\top}$$

5 Final remarks

This work supplies the need for a more rigorous, accessible and verified presentation of the proof of the axiomatizability of the fragments of Classical Logic induced by the clones located at the finite section of Post's lattice, which was originally given by Wolfgang Rautenberg in a paper with many typographic errors, with difficult notation and with many details missing. With the present study, it is expected that most doubts caused by such presentation problems regarding the veracity of this result disappear. This is a contribution that aids in the understanding of and provides more confidence to studies that apply this result somehow, with emphasis on those in the field of combination of logics, for which the properties of Hilbert-style proof systems are of special interest. Finally, the present study is a source of examples and a guide for the application of the Lindenbaum-Asser extension to proving the completeness of a Hilbert calculus with respect to a logical matrix, as well as for the verification, using the Lean theorem prover, of the derivability of rules in the calculus.

Further studies on the topic of the axiomatizability of fragments of Classical Logic are the verification of the completeness proof for the calculus $\mathscr{B}_{\mathsf{pt},\bot}$, or the proposal of another axiomatization for $\mathbb{2}_{\mathsf{pt},\bot}$; the search for simpler and more user-friendly calculi adequate for some fragments, like $\mathcal{B}_{\mathsf{ad}}$ (twenty-five rules in $\mathscr{B}_{\mathsf{ad}}$) and $\mathcal{B}_{\mathsf{pt,dc}}$ (twenty-one rules in $\mathscr{B}_{\mathsf{pt,dc}}$, some of them pretty complex); an analysis of the axiomatizability of the fragments of first-order Classical Logic; the investigation of the rules of interaction needed to produce an adequate calculus from the merging of two other arbitrary calculi, aiming to implement an optimized procedure that delivers axiomatizations for the fragment of Classical Logic corresponding to the combined language; and the search for a method with the purpose of, given a 2-matrix $\mathbb{2}_{\Sigma}$ whose signature is not previously known, producing an axiomatization over the same signature Σ for such matrix, a generalization of the procedure implemented in [7] based on Rautenberg's work.

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